Math Physics 210C 2000, June 16. Differential equations. Open book. No notes. GOOD LUCK!

1. Prove that $c_2 \ln|x|$ is a fundamental solution for $\triangle$ in two space dimensions with the appropriate choice of c_2 and give the constant c_2.

2. Find the solution to the initial value problem for the Heat equation

$$\begin{aligned} \partial_t u - \triangle u = F, \qquad t > 0, \quad -\infty < x < \infty \\ u(0,x) = g(x) \end{aligned}$$

3. Solve the equation

$$f(x) = x + \lambda \int_0^\infty e^{-(x+y)} f(y)\,dy$$

4. Let $0 \le c \le 1$ and consider the anharmonic oscillator

$$L_c = -\frac{d^2}{dx^2} + x^2 + cx^4,$$

acting on functions $f \in L^2(\mathbf{R}) \cap C^2$ that vanishes at infinity.
Note that if $c = 0$ then this is just the harmonic oscillator with eigenfunctions $v_n = H_n(x)e^{-x^2/2}$ where $H_n(x)$ are the Hermitte Polynomials ($H_1 = c_0$, $H_2 = c_2 x$ etc. see the book).

(a) Prove that the operator L_c is self adjoint, i.e. $(L_c u, v) = (u, L_c v)$, and positive, i.e. $(L_c u, u) \ge 0$ for $u \in D$.

(b) L_c has a discrete spectrum of eigenvalues $\lambda_n = \lambda_n(c)$, $\lambda_1 \le \lambda_2 \le ... \le \lambda_n \to \infty$. Give a reason why.

(c) Prove that the lowest eigenvalue $\lambda_1 \ge 1$ for all $0 \le c \le 1$. Hint compare L_c with an operators you know the eigenfunctions for.

(d) Use the min-max principle;

$$\lambda_n = \sup_{v_1,\dots,v_{n-1}} \quad \inf_{(u,v_1)=\dots=(u,v_{n-1})=0} (L_c u, u)/(u,u)$$

to find an lower bound for λ_2.