Math 210B Homework Problem set 1: Fundamental Solutions.

1. a) Show that $G(t,x)=\theta(t-|x|)/2$ is a fundamental solution for $\Box=\partial_t^2-\triangle$ in one space dimension. (Here $\theta(t)=1$ when $t>0$ and $\theta(t)=0$ when $t\leq 0$.)

b) Use the fundamental solution to find a nice formula for the solution of

$$\begin{aligned} \partial_t^2 u-\triangle u=F, \qquad & t>0, \quad -\infty<x<\infty \\ u(0,x)=f(x) \qquad & u_t(0,x)=g(x) \end{aligned}$$

In order to use the fundamental solution you need to rewrite the equation in a form with only a right hand hand side and vanishing initial conditions. This is done by applying $\Box$ to $u_0(t,x)=u(t,x)\theta(t)$.

2. Use the Fourier transform method to solve the problem

$$\begin{aligned} \partial_t^2 u-\triangle u=F, \qquad & t>0, \quad -\infty<x<\infty \\ u(0,x)=f(x) \qquad & u_t(0,x)=g(x) \end{aligned}$$

First take the Fourier transform with respect to x only; $\hat{u}(t,x)=\int e^{-ix\cdot\xi}u(t,x)\,dx$ and $\hat{F}(t,x)=\int e^{-ix\cdot\xi}u(t,x)\,dx$. Then the equation becomes an ordinary differential equation in t which can be solved using the fundamental solution (for the ode). Finally, take the inverse Fourier transform, and check that you get the same answer as in problem 1.

3. a) Give a proof of the divergence theorem.

b) Prove that $c_2/|x|$ is a fundamental solution for $\triangle$ in three space dimensions with the appropriate choice of c_2 and give the constant c_2.

4. Find the solution to the initial value problem for the Schrödinger equation

$$\begin{aligned} \partial_t u+i\triangle u=F, \qquad & t>0, \quad -\infty<x<\infty \\ u(0,x)=g(x) \qquad & \end{aligned}$$

Use the same method as in Problem 2.