Lecture 12: Wave Equations. Section 2.4.2,2.4.3 and:

The fundamental solution for the wave equation

$$\Box E = \delta(t,x), \quad \Box = \partial_t^2 - \triangle, \qquad (t,x) \in \mathbf{R}^{1+n}$$

is not hard to derive from the symmetries as well. Since $\Box$ is invariant under Lorentz transformations we expect the fundamental solution $E(t,x)$ to be invariant under Lorentz transformations as well, which means that it should be of the form $E(t,x) = f(t^2 - |x|^2)$, where f is a distribution. Plugging this into the equation gives after some calculation

$$4\rho f''(\rho) + 2(1+n)f'(\rho) = 0, \quad \rho = t^2 - |x|^2 \tag{12.1}$$

when $(t,x) \neq (0,0)$. This has the solution

$$\begin{cases} f(\rho) = & c_1 H(\rho), \quad \text{if} \quad n = 1 \\ f(\rho) = & c_2 H(\rho)\rho^{-1/2}, \quad \text{if} \quad n = 2 \\ f(\rho) = & c_3 \delta(\rho), \quad \text{if} \quad n = 3 \end{cases} \tag{12.2}$$

The constants can be calculated in the same way as we did for the fundamental solution of $\triangle$.

$$\begin{cases} E(t,x) = & c_1 H(t - |x|), \quad \text{if} \quad n = 1 \\ E(t,x) = & c_2 H(t - |x|)(t^2 - |x|^2)^{-1/2}, \quad \text{if} \quad n = 2 \\ E(t,x) = & c_3 \delta(t^2 - |x|^2) H(t), \quad \text{if} \quad n = 3 \end{cases} \tag{12.3}$$

Problem 12.2: Show in each case that (12.2) is a solution of (12.1).
Problem 12.3: If $n = 3$ prove that

$$\delta(t^2 - |x|^2)H(t) = \delta(t - |x|)/2|x| \tag{12.4}$$

Problem 12.4 Prove that $E(t,x)$ given above are fundamental solutions of $\Box$ and find the constants c_n.

Using the fundamental solution E for $\Box$ we can now solve the Cauchy problem

$$\Box u(t,x) = F$$
$$u(0,x) = f(x), \quad u_t(0,x) = g(x)$$

In fact let $u_0(t,x) = u(t,x)H(t)$ and $F_0(t,x) = F(t,x) = H(t)$ then

$$\Box u_0(t,x) = \Box u(t,x)H(t) = (\Box u(t,x))H(t) + 2u_t(t,x)\delta(t) + u(t,x)\delta'(t)$$
$$= F(t,x)H(t) + 2u_t(0,x)\delta(t) + u(0,x)\delta'(t) - u_t(0,x)\delta(t) = F_0(t,x) + g(x)\delta(t) + f(x)\delta'(t)$$

and hence

$$u_0(t,x) = E*\big(F_0(t,x) + g(x)\delta(t) + f(x)\delta'(t)\big) = E*F_0 + E*(g(x)\delta(t)) + \partial_t E*(f(x)\delta(t))$$

Let us now derive the solution formula if $n = 3$ in which case $E(t,x) = \delta(t - |x|)/4\pi|x|$ and hence
Problem 12.5 Show that
(12.5)

$$E * F_0(t,x) = \int\int F_0(t-s, x-y)\, \delta(s - |y|) \frac{1}{4\pi|y|} dy ds = \int_{|y| \le t} \frac{F_0(t - |y|, x - y)}{4\pi|y|} dy$$

and that

$$E * (g(x)\delta(t)) = t \int_{\omega \in S^2} \frac{g(x - t\omega)}{4\pi} dS(\omega)$$

where $dS(\omega)$ is the surface measure on the sphere S^2.