Lecture 16: Hyperbolicity. We went over Gårding's condition of hyperbolicity for wellposedness of the Initial value problem. We mostly followed the description from John 'Partial differential equations' section 5.2.a and 5.2.b, but the same thing can also be found in Evan's section 7.3.3. Here is the text from John:

Problems 3 from John's book in section 5.2.b

(a) Prove $E(\lambda)=\text{const.}$ when $\square u=0$. [Hint: Integrate $u_t \square u$ over the lens-shaped region $0<t<\lambda\phi(x)$.]
(b) Show that Q_λ as a quadratic form in $u_t, u_{x_1}, \ldots, u_{x_n}$ is positive definite, when S_λ is spacelike.
(c) Show that the initial data on S_0 of a solution of $\square u=0$ uniquely determine u on all S_λ with sufficiently small λ. (Compare Holmgren's theorem, p. 85.)

2. Let u be a solution of (1.68a, b, c) where $f=h=w=0$. Find the domain of dependence of $u(x,t)$ on g.

3. Consider the mixed problem for $u(x,t)=u(x_1,x_2,x_3,t)$

$$\square u=0 \quad \text{for } x_3>0,\ t>0 \tag{1.75a}$$

$$u=f(x), \qquad u_t=g(x) \quad \text{for } x_3>0,\ t=0 \tag{1.75b}$$

$$Mu=0 \quad \text{for } x_3=0,\ t>0, \tag{1.75c}$$

where M denotes a first-order operator of the form

$$M=\frac{\partial}{\partial t}+\sum_i \alpha_i \frac{\partial}{\partial x_i} \tag{1.75d}$$

with constant coefficients α_i, and f,g vanish for all sufficiently small $x_3>0$. Prove there exists a solution u provided $\alpha_3 \leqslant 0$. (Compare problem 2, p. 45.) [Hint: First determine $v=Mu$ for $x_3>0$, $t>0$ from its initial and boundary conditions as a solution of $\square v=0$. Next find u for $x_3>0$, $t>0$ as a solution of $Mu=v$ with initial condition $u=f$ by the methods of Chapter 1. Verify that the u obtained satisfies (1.75a, b, c).]

2. Higher-Order Hyperbolic Equations with Constant Coefficients

(a) *Standard form of the initial-value problem*

For functions $u(x,t)=u(x_1,\ldots,x_n,t)$ we define the differentiation operators

$$D=(D_1,\ldots,D_n)=\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right), \qquad \tau=\frac{\partial}{\partial t} \tag{2.1}$$

where D is the gradient vector with respect to the space variables. Using the Schwartz notation of Chapter 3 we can write the most general mth-order linear partial differential equation with constant coefficients in the form

$$P(D,\tau)u=w(x,t), \tag{2.2}$$

where $P(D,\tau)=P(D_1,\ldots,D_n,\tau)$ is a polynomial of degree m in its $n+1$ arguments. We associate with equation (2.2) in the half space $t>0$ the *initial conditions*

$$\tau^k u=f_k(x) \quad \text{for } k=0,\ldots,m-1 \text{ and } t=0. \tag{2.3}$$

We shall assume that the plane $t=0$ is noncharacteristic. This means that the coefficient $P(0,1)$ of τ^m in the polynomial P does not vanish. Dividing by a suitable constant we can bring about that

$$P(0,1)=1. \tag{2.4}$$

Problem (2.2), (2.3) for general data w, f_k can be reduced to the *standard problem* where the data have the special form

$$w=f_0=f_1=\cdots=f_{m-2}=0, \qquad f_{m-1}=g(x). \tag{2.5}$$

The solution of the standard problem (unique by Holmgren's theorem) will be denoted by $u_g(x,t)$. To achieve this reduction we first find a solution u of (2.2) with zero initial data. Such a solution is furnished according to *Duhamel's principle* by the formula

$$u(x,t)=\int_0^t U(x,t,s)\,ds, \tag{2.6}$$

where $U(x,t,s)$ for each parameter value $s>0$ is the solution of the initial-value problem

$$P(D,\tau)U(x,t,s)=0 \quad \text{for } t\geqslant s \tag{2.7a}$$

$$\tau^k U(x,t,s)=0 \quad \text{for } k=0,\ldots,m-2 \text{ and } t=s \tag{2.7b}$$

$$\tau^{m-1}U(x,t,s)=w(x,s) \quad \text{for } t=s. \tag{2.7c}$$

That it solves (2.2) is easily verified, using (2.4). Here for each $s\geqslant 0$ the function $U(x,t,s)$ is found by solving a standard problem; in fact

$$U(x,t,s)=u_g(x,t-s) \quad \text{where} \quad g(x)=w(x,s). \tag{2.8}$$

It remains to reduce the solution of the homogeneous equation

$$P(D,\tau)u=0 \tag{2.9}$$

with general initial conditions (2.3) to standard problems. For that purpose we arrange the polynomial $P(D,\tau)$ according to powers of τ:

$$P(D,\tau)=\tau^m+P_1(D)\tau^{m-1}+\cdots+P_m(D), \tag{2.10}$$

where $P_k(D)$ is a polynomial of degree $\leqslant k$ in $D_1,\ldots,D_n$. Using the differential equation (2.9) one easily verifies that the solution u with initial data (2.3) is representable in terms of the standard problems associated with each individual f_k by the formula

$$\begin{aligned} u=u_{f_{m-1}}&+(\tau+P_1(D))u_{f_{m-2}}+(\tau^2+P_1(D)\tau+P_2(D))u_{f_{m-3}}\\ &+\cdots\\ &+(\tau^{m-1}+P_1(D)\tau^{m-2}+P_2(D)\tau^{m-3}+\cdots+P_{m-1}(D))u_{f_0}. \end{aligned} \tag{2.11}$$

As an example we have for the solution of the wave equation

$$(\tau^2-c^2\Delta)u=0$$

with initial values

$$u=f, \qquad u_t=g \quad \text{for } t=0$$

the formula

$$u=u_g+\tau u_f$$

in agreement with (1.14).

A *system* of N linear partial differential equations of order m for N functions $u_1,\ldots,u_N$ can also be written in the form (2.2), where now u stands for the column vector with components $u_1,\ldots,u_N$, and $P(D,\tau)$ is a square $N\times N$ matrix whose elements are polynomials of degree $\leqslant m$ in $D_1,\ldots,D_n,\tau$. The data w,f_k in (2.2), (2.3) are column vectors. The solution u_g of a standard problem corresponds to the data (2.5). For a noncharacteristic initial plane $t=0$ the matrix $P(0,1)$ is nondegenerate, and we can assume that

$$P(0,1)=I \tag{2.12}$$

is the unit matrix. The solution of (2.2) with zero initial data still is described by Duhamel's formulas (2.6), (2.7a, b, c), and thus reduced to standard problems as in (2.8). The reduction of general initial data to standard ones is achieved by a modification of (2.11) which reads

$$u=u_{f_{m-1}}+\left(\tau u_{f_{m-2}}+u_{P_1f_{m-2}}\right)+\cdots+\left(\tau^{m-1}u_{f_0}+\tau^{m-2}u_{P_1f_0}+\cdots+u_{P_{m-1}f_0}\right). \tag{2.13}$$

In what follows we shall only have to deal with the standard problem

$$P(D,\tau)u=0 \quad \text{for } t\geqslant 0 \tag{2.14a}$$

$$\tau^k u=0 \quad \text{for } k=0,\ldots,m-2 \text{ and } t=0 \tag{2.14b}$$

$$\tau^{m-1}u=g(x) \quad \text{for } t=0. \tag{2.14c}$$

We call the differential equation or system of equations (2.14a) *hyperbolic* (with respect to the plane $t=0$), if the initial-value problem (2.14a, b, c) has a solution $u(x,t)$ of class C^m, for all $g(x)\in C_0^s(\mathbb{R}^n)$, where s is sufficiently large.* We also say that the plane $t=0$ is *spacelike*.

PROBLEM

Verify that formulas (2.11), respectively (2.13), give the solution of the initial-value problem (2.3), (2.9).

(b) *Solution by Fourier transformation*†

Following Cauchy a formal solution of the standard problem (2.14a, b, c) can be obtained by Fourier transformation with respect to the space

* Using the finiteness of the domain of dependence of u on g (implied, e.g., by Holmgren's theorem), one can show that in the hyperbolic case the problem (2.14a, b, c) has a solution for $g\in C^s(\mathbb{R}^n)$, even without the assumption of compact support.

†([8]).

variables. It will be an actual solution if the integrals involved converge adequately. We associate with a function $g(x) \in C_0^s(\mathbb{R}^n)$ its *Fourier transform* $\hat{g}$, defined by

$$\hat{g}(\xi) = (2\pi)^{-n/2} \int e^{-ix\cdot\xi} g(x)\,dx \tag{2.15}$$

$(x\cdot\xi = x_1\xi_1 + \cdots + x_n\xi_n)$.* For $g \in C_0^s$ with sufficiently large s the reciprocal formula

$$g(x) = (2\pi)^{-n/2} \int e^{ix\cdot\xi} \hat{g}(\xi)\,d\xi \tag{2.16}$$

holds. We find from (2.15) by integration by parts for any $k = 1, 2, \ldots, n$ that

$$\begin{aligned} -i\xi_k \hat{g}(\xi) &= (2\pi)^{-n/2} \int D_k(e^{-ix\cdot\xi})\, g(x)\,dx \\ &= -(2\pi)^{-n/2} \int e^{-ix\cdot\xi} D_k g(x)\,dx. \end{aligned}$$

We write this fundamental identity as

$$i\xi_k \hat{g} = \widehat{D_k g}. \tag{2.17}$$

By repeated application we find more generally for $g \in C_0^s$ and any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \leqslant s$ that

$$(i\xi)^\alpha \hat{g} = \widehat{D^\alpha g}. \tag{2.18}$$

Thus differentiation for g is transformed into multiplication for $\hat{g}$.

Formula (2.18) permits us to show that $\hat{g}(\xi)$ decreases rapidly for $\xi \to \infty$ when s is large. Let $\xi = (\xi_1, \ldots, \xi_n)$, where, say,

$$|\xi_j| = \max_k |\xi_k|. \tag{2.19a}$$

Then

$$|\xi| = \sqrt{\sum_k \xi_k^2} \leqslant \sqrt{n}\, |\xi_j| \tag{2.19b}$$

$$\begin{aligned} (1 + |\xi|)^s &= \sum_{k=0}^{s} \binom{s}{k} |\xi|^k \leqslant 2^s \sum_{k=0}^{s} n^{k/2} |\xi_j|^k \\ &\leqslant 2^s n^{s/2} \sum_{|\alpha| \leqslant s} |\xi^\alpha|. \end{aligned} \tag{2.19c}$$

* Here, of course, $i = \sqrt{-1}$. Observe that generally $\hat{g}$ is complex valued, even when the variables x, ξ, g are restricted to real values. In what follows the independent variables x, ξ will be assumed to be real, unless the contrary is stated, but $g, \hat{g}$, and the coefficients of the polynomial P will be allowed to be complex valued.

Consequently by (2.18), (2.15),

$$(1+|\xi|)^s|\hat{g}(\xi)| \leqslant 2^s n^{s/2} \sum_{|\alpha|\leqslant s} |(i\xi)^\alpha \hat{g}(\xi)|$$

$$\leqslant 2^s n^{s/2} \sum_{|\alpha|\leqslant s} \int |D^\alpha g(x)|\,dx \leqslant M_s < \infty, \tag{2.20}$$

where M_s depends on n, s, the size of the support of g, and the maxima of the absolute values of the derivatives of g of orders $\leqslant s$. It follows in particular that

$$|\hat{g}(\xi)| \leqslant \frac{M_{n+1}}{(1+|\xi|)^{n+1}} \quad \text{for } g\in C^{n+1}(\mathbb{R}^n) \tag{2.21}$$

and hence that the integral in (2.16) converges absolutely. Formula (2.16) is valid for $s>n$.

Let now $u(x,t)$ be a solution of (2.14a, b, c). To begin with we work with a *single* partial differential equation, so that u is a scalar. We write tentatively

$$u(x,t)=(2\pi)^{-n/2}\int e^{ix\cdot\xi}\hat{u}(\xi,t)\,d\xi, \tag{2.22}$$

where $\hat{u}(\xi,t)$ is the Fourier transform of u with respect to x. Purely formally we obtain by differentiation

$$0=P(D,\tau)u(x,t)=(2\pi)^{-n/2}\int e^{ix\cdot\xi}P(i\xi,\tau)\hat{u}(\xi,t)\,d\xi.$$

In addition for $t=0$

$$(2\pi)^{-n/2}\int e^{ix\cdot\xi}\tau^k\hat{u}(\xi,t)\,d\xi = \tau^k u(x,t)$$

$$=\begin{cases} 0 & \text{for } k=0,\dots,m-2 \\ g(x)=(2\pi)^{-n/2}\int e^{ix\cdot\xi}\hat{g}(\xi)\,d\xi & \text{for } k=m-1. \end{cases}$$

These relations are satisfied formally, when $\hat{u}(\xi,t)$ for each $\xi\in\mathbb{R}^n$ is a solution of the ordinary differential equation

$$P(i\xi,\tau)\hat{u}(\xi,t)=0 \qquad \left(\tau=\frac{d}{dt}\right) \tag{2.23a}$$

with initial values for $t=0$

$$\tau^k\hat{u}(\xi,t)=\begin{cases} 0 & \text{for } k=0,\dots,m-2 \\ \hat{g}(\xi) & \text{for } k=m-1. \end{cases} \tag{2.23b}$$

This leads to the *formal* solution* of (2.14a, b, c)

$$u(x,t)=(2\pi)^{-n/2}\int e^{ix\cdot\xi}Z(\xi,t)\,\hat{g}(\xi)\,d\xi, \tag{2.24a}$$

where Z as a function of t denotes the solution of the ordinary differential equation problem

$$P(i\xi,\tau)Z(\xi,t)=0 \tag{2.24b}$$

with initial values for $t=0$

$$\tau^k Z(\xi,t)=\begin{cases}0 & \text{for } k=0,\ldots,m-2\\ 1 & \text{for } k=m-1.\end{cases} \tag{2.24c}$$

There is no problem with the existence of Z. Moreover we can verify directly that the u given by (2.24a) is of class C^m in x,t for $x\in\mathbb{R}^n$, $t\geqslant 0$, and actually satisfies (2.14a, b, c), if $g\in C_0^s$ with $s>n$, and g and Z are such that all differentiations with respect to x or t of orders $\leqslant m$ can be carried out under the integral sign in (2.24a). This is certainly the çase when the resulting integrals converge absolutely. For that it is sufficient that the expressions

$$(1+|\xi|)^{n+1}|\tau^k\xi^\alpha Z(\xi,t)\,\hat{g}(\xi)| \quad \text{for } |\alpha|+k\leqslant m \tag{2.25}$$

are bounded uniformly in ξ,t for all $\xi\in\mathbb{R}^n$ and for t restricted to any finite interval $0\leqslant t\leqslant T$.

Of course, the expressions (2.25) will be bounded in any bounded set in ξt-space. What matters is only the behavior for large $|\xi|$. Here, to a certain extent, $\hat{g}(\xi)$ can be controlled by assuming that s is large enough, as is shown by the estimate (2.20). It is just a question of the growth of $Z(\xi,t)$ and its t-derivatives. If we can show that there exists a constant N, such that

$$|\tau^k Z(\xi,t)|\leqslant(1+|\xi|)^k N \quad \text{for all } \xi\in\mathbb{R}^n,\ 0\leqslant t\leqslant T,\ 0\leqslant k\leqslant m, \tag{2.26}$$

we find for the expressions (2.25), using (2.20), the upper bound

$$(1+|\xi|)^{n+1+m-s}NM_s.$$

For the boundedness of the expressions (2.25) it is here sufficient to assume that

$$s\geqslant n+1+m. \tag{2.27}$$

Formula (2.24a) will then represent a solution of our standard Cauchy problem (2.14a, b, c).

The proper condition on the partial differential equation (2.14a), i.e., on

* More precisely our arguments show that if there exists a solution $u(x,t)$ of (2.14a, b, c) of compact support in x and sufficiently often differentiable, then u must be given by the expression (2.24a).

the polynomial P, under which an estimate of the form (2.26) holds, and hence the initial-value problem can be solved, is:

Gårding's hyperbolicity condition. Equation (2.14a) is hyperbolic if there exists a real number c such that

$$P(i\xi, i\lambda) \neq 0 \text{ for all } \xi \in \mathbb{R}^n \text{ and all complex } \lambda \text{ with } Im\lambda \leqslant -c. \tag{2.28}$$

Condition (2.28) is equivalent to the statement that all of the m roots λ of

$$P(i\xi, i\lambda) = 0 \tag{2.29}$$

lie in one and the same half plane

$$\text{Im}\lambda > -c \tag{2.30}$$

of the complex number plane for all real vectors ξ.*

To establish the *sufficiency* of Gårding's condition we represent the solution Z of (2.24b, c) by a Cauchy integral:

$$Z(\xi, t) = \frac{1}{2\pi} \int_\Gamma \frac{e^{i\lambda t}}{P(i\xi, i\lambda)} d\lambda, \tag{2.31}$$

where the closed path of integration Γ runs around each root λ of (2.29) once in the counterclockwise direction. Indeed differentiation of Z as defined by (2.31) with respect to t results in multiplying the integrand by $i\lambda$ so that

$$\begin{aligned} P(i\xi, \tau)Z &= \frac{1}{2\pi} \int_\Gamma P(i\xi, i\lambda) \frac{e^{i\lambda t}}{P(i\xi, i\lambda)} d\lambda \\ &= \frac{1}{2\pi} \int_\Gamma e^{i\lambda t} d\lambda = 0 \end{aligned}$$

by Cauchy's theorem, while for $t = 0$ by (2.10)

$$\tau^k Z = \frac{1}{2\pi} \int_\Gamma \frac{i^k \lambda^k}{i^m \lambda^m + P_1(i\xi) i^{m-1} \lambda^{m-1} + \ldots + P_m(i\xi)} d\lambda.$$

Expanding Γ to infinity we see that this expression has the value 0 for $k = 0, \ldots, m-2$, and the value 1 for $k = m-1$.

We first derive an upper bound for the roots λ of (2.29). Using the expansion (2.10) we have

$$P(i\xi, i\lambda) = (i\lambda)^m + P_1(i\xi)(i\lambda)^{m-1} + \cdots + P_m(i\xi) = 0 \tag{2.32}$$

* Gårding showed that his condition (2.28) is necessary as well as sufficient. An even stronger statement holds in the case where the polynomial $P(D, \tau)$ is irreducible (i.e., not representable as product of lower degree polynomials): If the equation (2.14a) is not hyperbolic, the initial-value problem (2.14a, b, c) for $g \in C_0^\infty(\mathbb{R}^n)$ *never* has a solution, unless g vanishes identically.

For each kth-degree polynomial P_k we have a trivial estimate

$$|P_k(i\xi)| \leqslant M(1+|\xi|)^k \quad \text{for all } \xi \in \mathbb{R}^n \tag{2.33}$$

with a suitable constant M. Then for a root λ of (2.29)

$$|\lambda|^m \leqslant M \sum_{k=1}^{m} (1+|\xi|)^k |\lambda|^{m-k}.$$

Setting $\theta = |\lambda|/(1+|\xi|)$, we have then

$$\theta^m \leqslant M(1+\theta+\theta^2+\cdots+\theta^{m-1}).$$

This implies that either $\theta < 1$ or $\theta^m \leqslant Mm\theta^{m-1}$ and hence $|\theta| < Mm$. Thus for the roots λ of (2.29)

$$\theta = \frac{|\lambda|}{1+|\xi|} < 1 + Mm. \tag{2.34}$$

Denote by $\lambda_k(\xi)$ for $k = 1, \ldots, m$ the m (not necessarily distinct) roots λ of (2.29) taken in any order. Then

$$P(i\xi, i\lambda) = i^m \prod_{k=1}^{m} (\lambda - \lambda_k(\xi)). \tag{2.35}$$

Take for each $k = 1, \ldots, m$ the open disk of center λ_k and radius 1 in the complex λ-plane. Let U denote the union of these m (possibly overlapping) disks. Take for the path of integration Γ in (2.31) the boundary of U, which possibly consists of several closed curves and is composed of pieces of the boundaries of the individual unit disks. Then Γ runs once around each of the λ_k and has total length $\leqslant 2m\pi$. Moreover each of the points λ of Γ has distance $\geqslant 1$ from each of the λ_k, so that by (2.35)

$$|P(i\xi, i\lambda)| \geqslant 1 \quad \text{for } \lambda \in \Gamma.$$

Since each point of Γ has distance 1 from some λ_k we have from (2.34) and the Gårding condition (2.30)

$$\operatorname{Im}\lambda \geqslant -c-1, \qquad |\lambda| \leqslant 1+(1+Mm)(1+|\xi|) \leqslant (2+Mm)(1+|\xi|) \quad \text{on } \Gamma.$$

Thus

$$|e^{i\lambda t}| \leqslant e^{(1+c)t} \quad \text{for } t \geqslant 0, \lambda \in \Gamma.$$

It follows from (2.31) that

$$|\tau^k Z(\xi,t)| = \left| \frac{1}{2\pi} \int \frac{(i\lambda)^k e^{i\lambda t}}{P(i\xi, i\lambda)} \, d\lambda \right|$$

$$\leqslant m(2+Mm)^k (1+|\xi|)^k e^{(1+c)T} \tag{2.35a}$$

for $0 \leqslant t \leqslant T$, $\xi \in \mathbb{R}^n$, $0 \leqslant k \leqslant m$. This is an estimate of the type (2.26). It follows that for $g \in C_0^{n+m+1}(\mathbb{R}^n)$ the initial-value problem has a solution of class C^m for $t \geqslant 0$, provided the Gårding condition (2.28) is satisfied.

The integral (2.31) for Z is easily evaluated by the calculus of residues, in the case where all the roots λ_k are distinct. One finds that then

$$Z(\xi,t)=\sum_{k=1}^{m}\frac{e^{i\lambda_k t}}{P_\tau(i\xi,i\lambda_k)}. \tag{2.36}$$

As an example consider the n-dimensional wave equation corresponding to the operator

$$P(D,\tau)=\square=\tau^2-c^2\sum_{k=1}^{n}D_k^2. \tag{2.37a}$$

Here

$$P(i\xi,i\lambda)=-(\lambda^2-c^2|\xi|^2) \tag{2.37b}$$

has the real roots

$$\lambda_1=c|\xi|, \qquad \lambda_2=-c|\xi| \tag{2.37c}$$

satisfying the Gårding condition. Then by (2.36)

$$Z(\xi,t)=\frac{\sin(c|\xi|t)}{c|\xi|}. \tag{2.37d}$$

Thus the standard problem for the wave equation has the solution

$$u(x,t)=(2\pi)^{-n/2}\int e^{ix\cdot\xi}\frac{\sin(c|\xi|t)}{c|\xi|}\,\hat{g}(\xi)\,d\xi \tag{2.37e}$$

for $g\in C_0^{n+3}(\mathbb{R}^n)$.

If the polynomial $P(D,\tau)$ is *homogeneous* of degree m in D and τ (as in equation (2.37a)), we have for every solution (ξ,λ) of (2.29) and every real s

$$P(s\xi,s\lambda)=0, \qquad \operatorname{Im}(s\lambda)=s\operatorname{Im}\lambda.$$

Here $s\operatorname{Im}\lambda$ can be bounded from below for all s only, if $\operatorname{Im}\lambda=0$. Thus the Gårding condition for homogeneous P is that all roots λ of the equation

$$P(i\xi,i\lambda)=i^mP(\xi,\lambda)=0 \tag{2.38}$$

are real for all real ξ.

In many cases hyperbolicity can be inferred from properties of the *principal part* of P alone. We arrange the terms in the polynomial according to their degree, writing

$$P(D,\tau)=p_m(D,\tau)+p_{m-1}(D,\tau)+\cdots+p_0(D,\tau), \tag{2.39}$$

where $p_k(D,\tau)$ is a form of degree k in D and τ. Here $p_m(D,\tau)$ is identical with the principal part of $P(D,\tau)$ as defined in Chapter 3. We shall prove:

For the Gårding condition for P to be satisfied it is necessary that all roots λ of

$$p_m(\xi,\lambda)=0 \tag{2.40}$$

are real for all real ξ (i.e., that p_m satisfies the condition); *a sufficient requirement which implies the Gårding condition for P is that all roots* λ *of* (2.40) *are real and distinct for all real* $\xi \neq 0$.

To prove this statement we apply the substitution

$$\xi = \rho\eta, \qquad \lambda = \rho\mu, \tag{2.41}$$

where $\rho = |\xi|$ and η is a unit vector. Then $P(i\xi, i\lambda) = 0$ goes over into the equation

$$p_m(\eta,\mu) + \frac{1}{i\rho} p_{m-1}(\eta,\mu) + \cdots + \frac{1}{(i\rho)^m} p_0(\eta,\mu) = 0 \tag{2.42}$$

for μ, depending on the parameters ρ, η. By (2.4) the coefficient of μ^m in (2.42) has the value 1. The coefficients of the powers of μ not contributed by the principal part tend to 0 for $\rho \to \infty$, since η is bounded. Using the fact that the roots of a polynomial with highest coefficient 1 depend continuously on the coefficients, we see that for $\rho \to \infty$ the roots μ of (2.42) will tend* to the roots of

$$p_m(\eta,\mu) = 0. \tag{2.43}$$

Let there exist for a certain η a root μ_0 of (2.43) with $\operatorname{Im} \mu_0 \neq 0$. Assume $\operatorname{Im} \mu_0 = -\gamma < 0$ (otherwise replace η by $-\eta$ and μ_0 by $-\mu_0$). Then there exist roots μ of (2.42) for all sufficiently large ρ for which $\operatorname{Im} \mu < -\gamma/2$ and hence roots λ of (2.29) for which $\operatorname{Im} \lambda < -\rho\gamma/2$. This contradicts (2.30) for large ρ. Thus necessary for (2.30) is that the roots μ of (2.43) are real for all real η with $|\eta| = 1$, and then also for all real η. Assume next that the roots μ are real and distinct for real $\eta \neq 0$, in particular for $|\eta| = 1$. We now use the fact that roots of a polynomial equation with highest coefficient one are differentiable (even analytic) functions of the coefficients in any region not containing multiple roots. They will be uniformly Lipschitz continuous in any compact subregion. For large ρ and $|\eta| = 1$ the coefficients of equation (2.42) for μ differ from those of equation (2.43) by terms of order $1/\rho$. Hence the difference of the roots μ of (2.42) from appropriate roots of (2.43) is of order $1/\rho$ uniformly for $|\eta| = 1$. Since (2.43) has real roots, it follows that the imaginary parts of the roots μ of (2.42) are of order $1/\rho$, and hence the imaginary parts of the roots λ of (2.29) are bounded uniformly for all sufficiently large $\rho = |\xi|$. By (2.34) λ and $\operatorname{Im} \lambda$ also are bounded for bounded $|\xi|$. Thus (2.30) follows.

We call P *strictly hyperbolic* when its principal part $p_m(\xi, \lambda) = 0$ has real distinct roots for $\xi \neq 0$ hyperbolic. We see that strict hyperbolicity implies hyperbolicity. Thus, for example, any equation of the form

$$u_{tt} = c^2 \Delta u + ku \tag{2.44}$$

is hyperbolic.

*More precisely in a given neighborhood of a root μ of (2.43) of multiplicity γ there lie precisely γ roots of (2.42) if ρ is sufficiently large.

Formula (2.24a) for the solution $u(x,t)$ of the standard initial-value problem makes use of the values of $\hat{g}(\xi)$, which by (2.15) depend on the values of the given function g at all points. Actually by Holmgren's theorem the domain of dependence of $u(x,t)$ on the values of g is known to be finite; equivalently initial data g of compact support lead to solutions $u(x,t)$ of compact support in x. This is not obvious from the expression (2.24a), but can be deduced for strictly hyperbolic P from a version of the *Paley–Wiener theorem.* This involves a shift in the integrations in (2.24a) to complex ξ. For this we require estimates for the functions Z and $\hat{g}$ for *complex* arguments $\xi+i\zeta$ and real $t\geqslant 0$, where ξ and ζ are real.

Assume that the function $g(x)$ belongs to $C_0^s(\mathbb{R}^n)$ where $s\geqslant n+m+1$, and that the support of $g(x)$ lies in a ball $|x|<a$. For the complex vector $\xi+i\zeta$ we define $|\xi+i\zeta|$ by

$$|\xi+i\zeta|^2=\sum_{k=1}^{n}|\xi_k+i\zeta_k|^2=|\xi|^2+|\zeta|^2. \tag{2.45}$$

Then as in (2.19c) and by the same arguments

$$(1+|\xi+i\zeta|)^s\leqslant 2^s n^{s/2}\sum_{|\alpha|\leqslant s}|(\xi+i\zeta)^\alpha|. \tag{2.46}$$

We conclude from (2.15), (2.18) in analogy to (2.20) that

$$\begin{aligned}(1+|\xi+i\zeta|)^s|\hat{g}(\xi+i\zeta)|&<2^s n^{s/2}\sum_{|\alpha|\leqslant s}\int_{|x|<a}|e^{-ix\cdot(\xi+i\zeta)}D^\alpha g(x)|\,dx\\ &\leqslant 2^s n^{s/2}e^{a|\zeta|}\sum_{|\alpha|\leqslant s}\int|D^\alpha g(x)|\,dx\leqslant e^{a|\zeta|}M_s,\end{aligned} \tag{2.47}$$

since for real x with $|x|<a$

$$|e^{-ix\cdot(\xi+i\zeta)}|=e^{x\cdot\zeta}\leqslant e^{|x|\,|\zeta|}\leqslant e^{a|\zeta|}. \tag{2.48}$$

We proceed to estimate

$$Z(\xi+i\zeta,t)=\frac{1}{2\pi}\int_\Gamma\frac{e^{i\lambda t}}{P(i(\xi+i\zeta),i\lambda)}\,d\lambda$$

using as path of integration Γ again the boundary of the union of the unit disks with centers at the roots λ_k of

$$P(i(\xi+i\zeta),i\lambda)=0. \tag{2.49}$$

It follows, as in (2.35a) for $k=0$ that

$$|Z(\xi+i\zeta,t)|\leqslant me^{(1+c)t}, \tag{2.50a}$$

where

$$c=-\min_k(\operatorname{Im}\lambda_k)\leqslant\max_k|\operatorname{Im}\lambda_k|. \tag{2.50b}$$

To estimate c we apply the substitution

$$\xi + i\zeta = \rho\eta, \qquad \lambda = \rho\mu, \tag{2.51}$$

where $\rho = |\xi + i\eta|$ and η is a complex vector with $|\eta| = 1$. For a root λ of (2.49) we obtain again equation (2.42) for μ. The coefficients in equation (2.42) differ from those in the equation

$$p_m(\eta, \mu) = 0 \tag{2.52}$$

by terms of order $1/\rho$. Since $\eta = (\xi + i\zeta)/\rho$, and $|\xi/\rho|, |\zeta/\rho| < 1$ the coefficients in the equation (2.52) differ from those in the equation

$$p_m(\xi/\rho, \mu) = 0 \tag{2.53}$$

by terms of order $|\zeta|/\rho$. Since the roots μ of (2.53) are real and distinct for $\xi/\rho \neq 0$, it follows for the roots μ of (2.42) that

$$\operatorname{Im} \mu = O\left(\frac{1 + |\zeta|}{\rho}\right)$$

as long as ξ/ρ is bounded away from zero, and hence as long as $|\zeta|/\rho < \frac{1}{2}$. Thus the roots λ of (2.49) satisfy

$$\operatorname{Im} \lambda = O(1 + |\zeta|). \tag{2.54}$$

Since also as in (2.34).

$$|\operatorname{Im} \lambda| \leqslant |\lambda| = O(1 + |\xi + i\zeta|) = O(1 + \rho) = O(1 + |\zeta|)$$

for $|\zeta|/\rho > \frac{1}{2}$, we see that (2.54) is valid for all $\xi + i\zeta$. Thus there exists a constant M such that for the roots $\lambda = \lambda_k$ of (2.49)

$$|\operatorname{Im} \lambda_k| \leqslant M(1 + |\zeta|),$$

and hence, using (2.50a,b), (2.47)

$$|Z(\xi + i\zeta), t)| \leqslant m e^{(1 + M + M|\zeta|)t}$$

$$|e^{ix\cdot(\xi + i\zeta)} Z(\xi + i\zeta, t)\, \hat{g}(\xi + i\zeta)| \leqslant \frac{m M_s e^{-x\cdot\zeta + t + Mt + (a + Mt)|\zeta|}}{(1 + |\xi|)^s}. \tag{2.55}$$

By Cauchy's theorem we can in (2.24a) shift the domain of integration from that of real ξ to $\xi + i\zeta$ with fixed ζ without changing the value of the integral, due to the decay of the integrand for large $|\xi|$. Choose now ζ to be of the form $\sigma x/|x|$, where $\sigma > 0$. It follows from (2.24a), (2.55) that

$$|u(x,t)| \leqslant (2\pi)^{-n/2} m M_s e^{t + Mt - \sigma(|x| - a - Mt)} \int (1 + |\xi|)^{-s} d\xi.$$

If here

$$|x| > a + Mt$$

it follows for $\sigma \to \infty$ that $u(x,t) = 0$. Hence u for each $t > 0$ has bounded support lying in the ball $|x| \leqslant a + Mt$. The constant M here represents an upper bound for the *speed of propagation* of disturbances.

So far we have dealt with the standard problem for a single scalar equation $P(D,\tau)u=0$. The case of a system of equations with constant coefficients requires only minor adjustments. If P is an $N\times N$ matrix satisfying (2.12) a formal solution of (2.14a, b, c) is again furnished by (2.24a), where now, however, $Z(\xi,t)$ is an $N\times N$ matrix given by

$$Z(\xi,t)=\frac{1}{2\pi}\int_\Gamma e^{i\lambda t}(P(i\xi,i\lambda))^{-1}\,d\lambda. \tag{2.56}$$

Here Γ has to be a path in the λ-plane enclosing all singularities of the matrix P^{-1}, that is, all of the mN roots λ_k of the equation

$$Q(i\xi,i\lambda)=\det P(i\xi,i\lambda)=0. \tag{2.57}$$

Gårding's hyperbolicity condition for systems states that there exists a constant c such that

$$\operatorname{Im}\lambda>-c$$

for all roots λ of (2.57) for all real vectors ξ.

J. Hadamard introduced the important distinction between *well-posed* (also called *correctly-set*) problems and those that are *ill posed* (*improperly posed, incorrectly set*). The distinction applies specially to problems where a "solution" u is to be found from "data" g. Well-posed problems are those for which

(a) u exists for "arbitrary" g.
(b) u is determined "uniquely" by g.
(c) u depends "continuously" on g.

Here the words in quotation marks are somewhat vague and require that the spaces of admitted functions u and of functions g are specified. Typically well-posed problems are the Dirichlet problem for the Laplace equation and the initial-value problem for a hyperbolic equation with constant coefficients* (see [33]).

The initial-value problem for the scalar equation $P(D,\tau)u=0$ certainly is ill-posed when the principal part p_m of P does not satisfy Gårding's condition, that is, when there exists a real vector η and a nonreal scalar μ_0 such that

$$p_m(\eta,\mu_0)=0.$$

We can assume here that

$$\operatorname{Im}\mu_0=-\gamma<0,\qquad |\eta|=1.$$

*In the latter problem one can specify that $u\in C^m$ for $x\in\mathbb{R}^n$, $t\geqslant 0$ and that g has uniformly bounded derivatives of orders $\leqslant s$ with s chosen sufficiently large. Then u depends continuously on g in the sense that the maximum of $|u|$ can be estimated in terms of the maxima of the $|D^\alpha g|$ for $|\alpha|\leqslant s$. More generally the Cauchy problem with data on a *space like* surface is well posed.

Consider for any s exponential solutions of the form

$$(1+|\xi|)^{-s-m}e^{i(x\cdot\xi+\lambda t)}, \tag{2.58}$$

where $P(i\xi,i\lambda)=0$ and s is an arbitrary integer. Take here

$$\xi=\rho\eta, \qquad \lambda=\rho\mu.$$

For sufficiently large ρ we can find a λ for which $|\mu-\mu_0|<\gamma/2$. Then

$$|\lambda|<\left(|\mu_0|+\tfrac{1}{2}\gamma\right)\rho, \qquad \operatorname{Im}\lambda<-\tfrac{1}{2}\gamma\rho,$$

so that for $t=0$ and $|\alpha|\leqslant s$, $0\leqslant k\leqslant m$

$$\begin{aligned}|D^{\alpha}\tau^{k}u|&=(1+\rho)^{-s-m}|\lambda|^{k}|\xi^{\alpha}|\\&\leqslant(1+\rho)^{-s-m}\left(|\mu_0|+\tfrac{1}{2}\gamma\right)^{k}\rho^{k+|\alpha|}\\&\leqslant\left(1+|\mu_0|+\tfrac{1}{2}\gamma\right)^{m},\end{aligned}$$

while

$$u(0,t)=(1+\rho)^{-s-m}|e^{i\lambda t}|\geqslant(1+\rho)^{-s-m}e^{\gamma\rho t/2}.$$

Thus the initial data and their derivatives of orders $\leqslant s$ are bounded uniformly for all x, while $u(0,t)\to\infty$ for $\rho\to\infty$ and any fixed $t>0$. Here u does not depend "continuously" on its initial data.

Problems

1. For $n=3$ identify the solution of the standard initial-value problem for the wave equation given by (2.37e) with the solution $u=tM_g(x,ct)$ obtained from (1.14). [Hint: Compute $M_g(x,ct)$ in terms of $\hat{g}$ from (2.16).]

2. Solve the standard initial-value problem for the system of equations of elastic waves

$$\rho\frac{\partial^2u_i}{\partial t^2}=\mu\Delta u_i+(\lambda+\mu)\frac{\partial}{\partial x_i}\left(\sum_i\frac{\partial u_k}{\partial x_k}\right) \tag{2.59}$$

(with positive constants ρ,λ,μ) in the form (2.24a), computing the matrix $Z(\xi,t)$ explicitly from (2.56). [Answer: $Z(\xi,t)$ is the matrix with elements

$$\frac{c_1(\delta_{ik}|\xi|^2-\xi_i\xi_k)\sin(c_2|\xi|t)+c_2\xi_i\xi_k\sin(c_1|\xi|t)}{c_1c_2|\xi|^3}, \tag{2.60}$$

where $c_1^2=(\lambda+2\mu)/\rho$, $c_2^2=\mu/\rho$.]

3. Show that for $n=1$, $m=1$ and any N the system of equations

$$u_t+Bu_x-Cu=0$$

is strictly hyperbolic, when the matrix B has real and distinct eigenvalues. (Compare (5.12) of Chapter 2.)

4. Prove that, when Gårding's condition is satisfied, the solution u of (2.14a, b, c) can be written

$$u(x,t)=(1-\Delta_x)^s \int K(x-y,t)\,g(y)\,dy, \tag{2.61a}$$

where

$$K(x-y,t)=(2\pi)^{-n/2}\int e^{i(x-y)\cdot\xi}(1+|\xi|^2)^{-s}Z(\xi,t)\,d\xi \tag{2.61b}$$

and s is any integer exceeding $n/2$. [Hint: Introducing $h(x)=(1-\Delta_x)^s g(x)$, we can substitute for $\hat{g}$ in (2.24a) the expression $(1+|\xi|^2)^{-s}\hat{h}(\xi)$. Interchanging the integrations yields

$$u(x,t)=\int K(x-y)h(y)\,dy$$

from which (2.61a) can be derived.]

(c) *Solution of a mixed problem by Fourier transformation*

In many cases mixed initial-boundary-value problems can be solved by Fourier transformation, when the domain of the solution is a half space. The method will be illustrated (following R. Hersh*) by a problem for the wave equation for $n=3$, which could also be solved by reflection. (Compare problem 3, p. 143.) We seek to find a $u(x,t)=u(x_1,x_2,x_3,t)$ for which

$$\square u=0 \quad \text{for } x_3 \geqslant 0,\ t>0 \tag{2.62a}$$

$$u=u_t=0 \quad \text{for } x_3>0,\ t=0 \tag{2.62b}$$

$$Mu=u_t+\alpha_1 u_{x_1}+\alpha_2 u_{x_2}+\alpha_3 u_{x_3}=h(x_1,x_2,t) \quad \text{for } x_3=0,\ t>0. \tag{2.62c}$$

We assume here that the α_k are constant, that $\alpha_3<0$, and (to avoid inconsistencies for $t=x_3=0$) that there exists a positive ε such that

$$h(x_1,x_2,t)=0 \quad \text{for } t<\varepsilon. \tag{2.62d}$$

Let moreover $h\in C_0^s(\mathbb{R}^3)$ with a sufficiently large s.

The building blocks are again exponential solutions

$$u=e^{i(\lambda t+\xi_1 x_1+\xi_2 x_2+\xi_3 x_3)} \tag{2.63a}$$

of (2.62a), for which the relation

$$\lambda^2-c^2(\xi_1^2+\xi_2^2+\xi_3^2)=0 \tag{2.63b}$$

will have to hold. For these u we have

$$Mu=i(\lambda+\alpha_1\xi_1+\alpha_2\xi_2+\alpha_3\xi_3)e^{i(\lambda t+\xi_1 x_1+\xi_2 x_2)} \quad \text{for } x_3=0. \tag{2.63c}$$

This leads to a formal solution of (2.62a,c) given by

$$u(x,t)=(2\pi)^{-3/2}\int \frac{e^{i(\lambda t+\xi_1 x_1+\xi_2 x_2+\xi_3 x_3)}}{i(\lambda+\alpha_1\xi_1+\alpha_2\xi_2+\alpha_3\xi_3)}\hat{h}(\xi_1,\xi_2,\lambda)\,d\xi_1\,d\xi_2\,d\lambda. \tag{2.64}$$

* Mixed problems in several variables, *J. Math. Mech.* **12** (1963), 317–334.