Lecture 5: Distributions. Let $C_0^\infty(\mathbf{R}^n)$ (or $\mathcal{D}$) denote the set of infinitely differentiable functions that have compact support. (The support of a continuous function ϕ is defined by supp $\phi = \{x \in \mathbf{R}^n;\ \phi(x) \neq 0\}$.) The seminorms

$$\rho_{\alpha,K}(\phi) = \sup_{x\in K} |\partial^\alpha \phi(x)|,$$

where K is any compact subset, makes C_0^∞ into a topological space, a Freche' space. We say that $\phi_n \to \phi$ in C_0^∞ if ϕ and all ϕ_j's are supported in a fixed compact set K and $\rho_{\alpha,K}(\phi_n - \phi) \to 0$, as $n \to \infty$ for every fixed α.
A linear map $L : C_0^\infty \to \mathbf{C}$ is called continuous if $\phi_n \to \phi$ implies that $L(\phi_n) \to L(\phi)$. If L is linear it is equivalent to only assume continuity at $\phi = 0$.

Definition. Let $\mathcal{D}'$ denote the dual space of C_0^∞, i.e. the space of all continuous linear functionals : $C_0^\infty \to \mathbf{C}$. $\mathcal{D}'$ is called the space of distributions.

The continuity is equivalent to that for every compact K there are C, N so that

$$|L(\phi)| \le C \sum_{|\alpha|\le N} \sup_{x\in K} |\partial^\alpha \phi(x)|, \qquad \text{if} \qquad \operatorname{supp}\phi \subset K.$$

If f is a distribution we will write $\langle f, \phi\rangle$ for what we just called $L(\phi)$.

A bounded function can be viewed as a distribution given by $\langle f, \phi\rangle = \int f\phi\, dx$. In fact, if supp $\phi \subset K$, then $|\int f\phi\, dx| \le \int_K |f|\, dx\, \sup_x |\phi(x)| \le C \sup_x |\phi(x)|$. Even if f is a distribution we will sometimes use $\int f\phi\, dx$ to denote $\langle f, \phi\rangle$, keeping in mind that it is to be interpreted as a linear functional and not an usual integral.

Moreover, any derivative of a function is a distribution even if the function is not differentiable in the usual sense. In fact, the main motivation to introduce distributions is to generalize the concept of derivative. We define the derivative by

$$\int (\partial^\alpha f)\, \phi\, dx = (-1)^{|\alpha|} \int f\, \partial^\alpha \phi\, dx$$

This defines a distribution that agrees with the usual derivative if f is smooth by integrating by parts.

Any weak limit of a distribution is a distribution. We say that $f_n \to f$ weakly if

$$\int f_n \phi\, dx \to \int f\phi\, dx, \quad \phi \in C_0^\infty.$$

Moreover, it follows directly from the definitions that $\partial^\alpha f_n \to \partial^\alpha f$ if $f_n \to f$.

Any distribution f is the weak limit of a sequence of $f_n \in C_0^\infty$.
In fact, let $\phi \in C_0^\infty$ satisfy $\int \phi\, dx = 1$ and set $\phi_\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon^n$. The convolution $f_k(x) = \phi_{1/k} * f(x) = \int f(y)\phi_{1/k}(x-y)\, dy = \langle f, \phi_{1/k}(x - \cdot)\rangle$ is well defined as the distribution f acting on the test function $\phi_{1/k}(x-y)$, considered as a function of y. That $f_k \in C^\infty$ is seen by looking at difference quotients $(f_k(x+he_j) - f_k(x))/h$ and using that $(\phi_{1/k}(x+he_j) - \phi_{1/k}(x))/h \to \partial_j \phi_{1/k}(x)$ in C_0^∞. To show that $f_k \to f$:

$$\langle f_k, \psi\rangle = \int\!\!\int f(y)\phi_{1/k}(x-y)\, dy\, \psi(x) dx = \int f(y) \int \phi_{1/k}(x-y)\psi(x)\, dx\, dy = \langle f, \phi_{1/k} * \psi\rangle$$

It follows from the proof of Lemma 3.2 that $\phi_{1/k} * \psi \to \psi$, and hence $f_k \to f$ weakly.

The delta function. The simplest example of a distribution which is not a function is the "delta function" at a $\delta_a(x) = \delta(x-a)$ defined by

$$\int \phi(x)\,\delta_a(x)\,dx = \phi(a), \quad \phi \in C_0^\infty$$

It is a distribution since it satisfies $|\langle \delta_a, \phi\rangle| \le \sup_x |\,\phi(x)|$.
Physically one can think of the delta function as a point charge; $\delta_a(x)=\infty$ when $x=a$ and $\delta_a(x)=0$ when $x\neq a$ in such a way that the total change is $\int \delta_a(x)\,dx=1$. One can think of it as a limit of a the sequence $\phi_\varepsilon(x)=\phi(x/\varepsilon)/\varepsilon^n$, with $\int \phi\,dx=1$. A third interpretation in one variable is as the derivative of the step function: Define the Heavyside function H by $H(x)=1$ for $x>0$ and $H(x)=0$ for $x<0$. Then in the sense of distributions $H'(x)=\delta(x)$. In fact

$$\int H'(x)\phi(x)\,dx = -\int H(x)\phi'(x)\,dx = -\int_0^\infty \phi'(x)\,dx = \phi(0), \quad \phi \in C_0^\infty.$$

Problem 5.1 Let $\psi(t) = e^{-1/t}$, when $t>0$, and $\psi(t)=0$, when $t \le 0$. Show that $\psi \in C^\infty(\mathbf{R})$. Let $\eta(x) = \psi(1-|x|^2)$, where $|x| = \sqrt{x_1^2+\cdots+x_n^2}$. Show that $\eta(x) \in C_0^\infty(\mathbf{R}^n)$.