Lecture 1: Introduction.

1.2 Space-time in pre-relativity and Special relativity.
One of the difficulties in understanding relativity is overcoming our misconceptions about space and time from Newtonian physics.
In Newtonian physics we have a concept of an absolute time and for each fixed time the universe is a 3 dimensional space of simultaneous events; given an event p, all other events are either in the future of p, in the past of p, or simultaneous to p.
However in special relativity we learn that no information can pass faster then the speed of light and hence there is a future light cone whose interior consist of events in the future of p and a past light cone whose interior consist of events in the past of p, but everything outside the light cone is not casually related to p.

In special relativity as well as in prerelativity physics there is a notion of inertial motion, i.e. the "nonaccelerating motion without subject to external forces.
A basic postulate is that physics should look the same for all inertial observers.
If we have inertial observers O and O', moving apart with constant velocity in the x direction, that label the same event by (t, x, y, z) and (t', x', y', z'), then

$$t' = t, \quad x' = x - vt, \quad y' = y, \quad z' = z,$$

in prerelativity whereas in relativity they are related by a Lorentz transformation

$$t' = (t - vx)/(1 - v^2)^{1/2}, \quad x' = (x - vt)/(1 - v^2)^{1/2}, \quad y' = y, \quad z' = z$$

where we have chosen units so the speed of light $c = 1$. In fact the Lorenz transformations are precisely the linear transformations that leave the light cones invariant, in other words the speed of light is the same in any inertial frame.

1.3 The spacetime metric.
In prerelativity physics the time interval between events $\triangle t$ as well the spacial interval $|\triangle \mathbf{x}|$ between simultaneous events have an absolute meaning.
In special relativity neither of these have absolute meaning and the only observer independent quantity is the Minkowski metric

$$-(\triangle t)^2 + (\triangle x)^2 + (\triangle y)^2 + (\triangle z)^2$$

In fact this is the only quadratic form invariant under Lorentz transformations.

1.4 General Relativity and gravity.
Newton's second law states that mass times acceleration is equal to the force:

$$m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{F}$$

Newton thought of gravity as a force, the force on a mass m by a larger mass M is

$$F = -\frac{GmM}{r^2}$$

where r is the distance between M and m.

Newton's theory of gravitation is not consistent with special relativity since it invokes the notion of instantaneous influence of one body on another.
Einstein's thought that since all bodies fall the same in a gravitational field it ought to be a property of spacetime itself. A freely falling body is the natural unforced state. It is the motion along a geodesic in the spacetime metric:

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0.$$

In the early 1800s Gauss asked how much of the geometry of a surface is independent of how it bends in space. Riemannian geometry is designed to describe the universe of creatures who live on a curved surface and who are unaware of space outside and can only measure distances and areas on the surface. This lead to the modern notion of a manifold independent of a surrounding space. Riemannian Geomtry is about manifolds with a notion of a distance called a metric, how this can be used to define curvatures and how the curvatures can characterize a manifold. Einstein realized that this theory could be used to describe how space curves under the influence of gravity which lead to the general theory of relativity. The metric is invariantly defined as a quadratic form on the tangent space. The Christoffel symbol or any combination of the first order derivatives of the metric are not invariantly defined as a tensor on the tangent space but there are combinations of second order derivatives with first order derivatives that form an invariant form on the tangent space. The Riemann curvature tensor is a 4 tensor and hence

$$R_{abc}{}^{d} = -\partial_a\Gamma^d_{bc} + \partial_b\Gamma^d_{ac} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{bc}\Gamma^d_{ae},$$

where the Christoffel symbols are given by

$$\Gamma^d_{ab} = g^{dc}\big(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}\big)/2$$

we see that $R_{abc}{}^{d} \sim \partial^2 g + (\partial g)^2$.

If we trace the Riemann curvature tensor $R_{abc}{}^{d}$ we get the Ricci curvature

$$R_{ac} = R_{adc}{}^{d}$$

and if we trace again the scalar curvature

$$R = R_{ac}g^{ac}$$

Einstein's vacuum equations without any forces from matter are

$$R_{ab} = 0.$$

We will motivate these later but for now let us just say that if we want an equation for the metric that is invariant under changes of coordinates, under changes of accelerating frame it has to be in terms of the curvature. Moreover since we expect physics to be such that a particles path in the absence of exterior forces is determined by its initial position and velocity, then it has to be a second order equation. Therefore it has to be an equation in terms of the curvature itself. If the Riemann curvature vanishes then the metric can be transformed to the flat

Minkowski metric by a change of coordinates. Just saying that the scalar curvature vanishes is too general so what is left is to say that the Ricci curvature vanishes.

In the presence of exterior forces the above equations have to be modified since since its not automatically divergence free. The Einstein tensor is defined by

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$$

It follows from the Biachi identity that

$$\nabla^a G_{ab} = 0, \qquad T^a = g^{ac} \nabla_c$$

Einstein's equations in the presence of exterior forces is

$$G_{ab} = T_{ab}$$

where T_{ab} is the energy momentum tensor of the matter fields that is expect to satisfy additional equations, in particular it has to be divergence free $\nabla^a T_{ab} = 0$.

We will now explain how matter influence the gravitational field or the metric of space time. Here Einstein was influenced by some ideas going under the name of *Mach's principle* but which also go back to others like Riemann. They felt that matter should contribute to the local definition of nonaccelerating and that in a universe with no matter there should be no meaning of these concepts.

How is space-time geometry going to be influenced by the matter distribution? In Newtonian theory the gravitational force is represented by the gradient of a gravitational potential. The gravitational potential satisfy Poisson's equation

$$\triangle \phi = 4\pi \rho$$

where ρ is the mass density. Consider two small masses m separated by a vector $\mathbf{x}$ influenced by the gravitational potential. Then the difference in force between them is $-\mathbf{x} \cdot \nabla\nabla\phi$ which is determining the relative acceleration. On the other hand the relative acceleration of two geodesics in curved space is given by the curvature $-R_{cbd}{}^a v^c x^b v^d$, where v^a is the 4-velocity of the particles and x is the deviation vector. This suggests the correspondence

$$R \sim \nabla^2 \phi.$$

Since $R \sim \partial^2 g$ this means that the metric is like the gravitational potential and that we should get an equation for the curvature. Also since

$$T_{ab} v^a v^b \sim \rho$$

this gives some idea of Einstein's equations

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}$$

Harmonic coordinates and the initial value problem Eisntein's equations are invariant under changes of coordinates. In harmonic coordinates they become a system of nonlinear wave equations for the metric:

$$\Box_g g_{\alpha\beta} = F_{\alpha\beta}(g, \partial g)$$

For a system of nonlinear wave equations we can locally solve the initial value problem, i.e. if we are given an initial manifold which corresponds to time equal to 0 with an initial metric and its time derivative then we can find a solution to Einstein's equations for small times (provided the initial data satisfies some constraint).

What are the problems in General Relativity?. Let M be four manifold with a Lorentzian metric $g_{\alpha\beta}$ of signature $(-1,1,1,1)$. Einstein's vacuum equations are that the Ricci curvature of the metric vanishes

$$R_{\mu\nu} = 0$$

The corresponding equations with matter are $R_{\mu\nu} = T_{\mu\nu}$ where $T_{\mu\nu}$ is the energy momentum tensor of the fields that satisfy additional equations. (This is to be compared to Newton's equations that mass times acceleration if the force $ma = F$.)

The initial value problem for Einstein's equations are that we are given a three manifold Σ_0 with a Riemanninan metric g^0 and a second fundamental form k^0, and we are supposed to find a four manifold M with a Lorantzian metric g satisfying the Einstein equations, and an imbedding of Σ in M such that the restriction of the metric g to Σ is g^0 and the second fundamental form of Σ in M is k^0. The initial value problem is over determined so g^0 and k^0 has to satisfy the constraint equations.

The study of Einstein's equations relate to many fields: Geometry, Partial Differential Equations, Numerical Relativity, Physics and Astronomy.

In Geometry/PDE the problem divides into studying the constraint equation, which are elliptic and the evolution equations which are hyperbolic. An example of a theorem for the constraint equations is the positive mass theorem of Schoen and Yau, which say that the metric $g^0_{\alpha\beta} \sim \delta_{\alpha\beta}(1 - M/r)$, where the mass of the universe $M > 0$.

An example of a theorem for the evolution problem is that Christodoulou-Klainerman proved the global stability of Minkowski space time, which says that Einstein's equations have global solutions if initial data are sufficiently close to that of Minkowski space time $m_{\mu\nu} = \text{diag}(-1,1,1,1)$. The local problem was solved by Chouque-Bruhat 50 years ago.

However, for large data singularities, called black holes, can form for Einstein's equations. There are the famous singularity theorems of Hawking and Ellis. Penrose came up with the so called cosmic censorship conjectures that states that for generic initial data the singularities are hidden by an even horizon which is such that no information can come out from the black hole side of the event horizon to the other side making it possible to cut that region out of space time. One can fall into the black hole but nothing in there can influence what goes on outside and hence not contradict uniqueness.

Penrose also conjectured the peeling estimates that say that the curvature has to decay at a certain rate to flat space. The radiation is important to study because this is what we can see.

On the Physics side we have people like Thorne at Caltech and Tamour at IHES. People are building a gravitational wave detector for \$200,000,000. Since the noise level is large they have to know what they are looking for and it is therefore important to do numerical simulations, with supercomputers. There are also people in astronomy departments studying cosmology.