Lecture 10: 4.3 General Relativity.

The *equivalence principle* states that all bodies fall the same way in a gravitational field. This is not the case in an electromagnetic field which only influences charged particles. Because of this Einstein came up with the idea that instead of thinking of the gravitation as a force that accelerates bodies one should think of it as a part of spacetime. The world line of freely falling bodies in a gravitational field are simply the geodesics of a curved spacetime.

Let u be the unit 4-velocity of a particle. A free particle travels along a geodesic

$$u^a \nabla_a u^b = 0$$

where ∇ is covariant differentiation with respect to the spacetime Lorentzian metric. On the other hand a particle of mass m and charge q in an electromagnetic field F_{ab} satisfies the Lorentz force equation:

$$u^a \nabla_a u^b = \frac{q}{m} F^b{}_c u^c$$

We define the energy momentum tensor for perfect fluid is almost the same as in flat spacetime;

$$T_{ab} = \rho\, u_a u_b + P(g_{ab} + u_a u_b).$$

The divergence free condition is now

$$\nabla^a T_{ab} = 0$$

which as before leads to the equations

$$u^a \nabla_a \rho + (\rho + P)\nabla^a u_a = 0$$
$$(P + \rho) u^a \nabla_a u_b + (g_{ab} + u_a u_b)\nabla^a P = 0$$

The curved equation for the scalar field is

$$\nabla^a \nabla_a \phi - m^2 \phi = 0.$$

Here $\nabla^a \nabla_a$ is the geometric wave operator in the metric g. The energy momentum tensor

$$T_{ab} = \nabla_a \phi\, \nabla_b \phi - \frac{1}{2} g_{ab} \big(\nabla^c \phi\, \nabla_c \phi + m^2 \phi \big)$$

satisfies $\nabla^a T_{ab} = 0$. Maxwell's equations take the form

$$\nabla^a F_{ab} = -4\pi j_b$$

$$\nabla_{[a} F_{bc]} = 0 = \nabla^a * F_{ab}$$

where j^a is the current 4-vector of electric charge. The energy momentum tensor for the *electromagnetic field* is

$$T_{ab} = \frac{1}{4\pi} \big(F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{de} F^{de} \big)$$

which as before satisfy $\nabla^a T_{ab} = 0$, if $j_a = 0$.

We have now given the equations of motions in curved spacetime showing how the metric or gravity influence the motion of mass. However, we also need to give an equation for how mass influences the metric or gravitational field.

We will now explain how matter influence the gravitational field or the metric of space time. Here Einstein was influenced by some ideas going under the name of *Mach's principle* but which also go back to others like Riemann. They felt that matter should contribute to the local definition of nonaccelerating and that in a universe with no matter there should be no meaning of these concepts.

How is spacetime geometry going to be influenced by the matter distribution? In Newtonian theory the gravitational force is represented by the gradient of a gravitational potential. The gravitational potential satisfy Poisson's equation

$$\triangle\phi = 4\pi\rho$$

where ρ is the mass density. Consider two small masses m separated by a vector $\mathbf{x}$ influenced by the gravitational potential. Then the difference in force between them is $-\mathbf{x}\cdot\nabla\nabla\phi$ which is determining the relative acceleration. On the other hand the relative acceleration of two geodesics in curved space is given by the curvature $-R_{cbd}{}^{a}v^{c}x^{b}v^{d}$, where v^{a} is the 4-velocity of the particles and x is the deviation vector. This suggests the correspondence

$$R \sim \partial^2\nabla^2\phi.$$

Since $R \sim \partial^2 g$ this means that the metric is like the gravitational potential and that we should get an equation for the curvature. Also since

$$T_{ab}v^{a}v^{b} \sim \rho$$

this gives some idea of Einstein's equations

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = 8\pi T_{ab}$$

10.2 Harmonic coordinates.

We will now show that Einstein's equations in the case of weak fields reduces are consistent with Newton's equations. We will assume that

$$g_{ab} = \eta_{ab} + \gamma_{ab}$$

where γ_{ab} is small compared to g_{ab}. Then modulo quadratic terms in h

$$g^{ab} = \eta^{ab} - \gamma^{ab} + O(\gamma^2), \qquad \gamma^{ab} = \eta^{ac}\eta^{bd}\gamma_{cd}$$

Two metric that differs by a diffeomorphism define the same spacetime so there is freedom of choice of representative within a diffeomorphism class. We choose to impose the harmonic coordinate condition on the metric

$$g^{ab}\Gamma_{ab}^{c} = 0,$$

where Γ_{ab}^{c} is the Christoffel symbol

$$\Gamma_{ab}^{c} = \frac{1}{2} g^{cd}\big(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\big).$$

In fact, given a metric one can always at least locally make a change of coordinates so in the new coordinates the harmonic coordinate condition hold. We just solve a system for the new coordinates $\Box_g x^d = 0$, where the geometric wave operator is

$$\Box_g \phi = g^{ab}\nabla_a\nabla_b\phi = g^{ab}\partial_a\partial_b\phi + g^{ab}\Gamma_{ab}^{c}\partial_c\phi$$

Since the geometric wave operator is invariant under changes of coordinates it must also vanish when expressed in the x^d coordinates $0 = \Box_g\, x^d = g^{ab}\partial_a\partial_b x^c + g^{ab}\Gamma_{ab}^{c}\partial_c x^d = g^{ab}\Gamma_{ab}^{d} = 0$, since $\partial_c x^d = \delta_c^d$. In the harmonic coordinates the geometric wave operator hence reduces to

$$\widetilde{\Box}_g \phi = g^{ab}\partial_a\partial_b\phi$$

Recall that

$$R_{\mu\nu\rho}{}^{\sigma} = \partial_\nu\Gamma_{\mu\rho}^{\sigma} - \partial_\mu\Gamma_{\nu\rho}^{\sigma} + \Gamma_{\mu\rho}^{\alpha}\Gamma_{\alpha\nu}^{\sigma} - \Gamma_{\nu\rho}^{\alpha}\Gamma_{\alpha\mu}^{\sigma},$$

Hence modulo terms that are quadratic in ∂h the Ricci curvature is

$$\begin{aligned}
R_{\mu\rho} = R_{\mu\nu\rho}{}^{\nu} &= \partial_\nu\Gamma_{\mu\rho}^{\nu} - \partial_\mu\Gamma_{\nu\rho}^{\nu} + \Gamma_{\mu\rho}^{\alpha}\Gamma_{\alpha\nu}^{\nu} - \Gamma_{\nu\rho}^{\alpha}\Gamma_{\alpha\mu}^{\nu}\\
&= \frac{1}{2}g^{\nu d}\partial_\nu\big(\partial_\mu g_{\rho d} + \partial_\rho g_{\mu d} - \partial_d g_{\mu\rho}\big) - \frac{1}{2}g^{\nu d}\partial_\mu\big(\partial_\nu g_{\rho d} + \partial_\rho g_{\nu d} - \partial_d g_{\nu\rho}\big) + O((\partial g)^2)\\
&= \frac{1}{2}g^{\nu d}\partial_\nu\big(\partial_\rho g_{\mu d} - \partial_d g_{\mu\rho}\big) - \frac{1}{2}g^{\nu d}\partial_\mu\big(\partial_\rho g_{\nu d} - \partial_d g_{\nu\rho}\big) + O((\partial g)^2)\\
&= -\frac{1}{2}g^{\nu d}\partial_\nu\partial_d g_{\mu\rho} + \frac{1}{2}g^{\nu d}\partial_\rho\big(\partial_\nu g_{d\mu} + \partial_d g_{\nu\mu} - \partial_\mu g_{\nu d}\big) + \frac{1}{2}g^{\nu d}\partial_\mu\big(\partial_\nu g_{d\rho} + \partial_d g_{\nu\rho} - \partial_\rho g_{\nu d}\big) + O((\partial g)^2)\\
&= -\frac{1}{2}g^{\nu d}\partial_\nu\partial_d g_{\mu\rho} + \frac{1}{2}g^{\nu d}\partial_\rho\big(g_{\mu c}\Gamma_{\nu d}^{c}\big) + \frac{1}{2}g^{\nu d}\partial_\mu\big(g_{\rho c}\Gamma_{\nu d}^{c}\big) + O((\partial g)^2)
\end{aligned}$$

If the metric satisfy the harmonic coordinate condition $g^{\nu d}\Gamma_{\nu d}^{c} = 0$ then

$$R_{\mu\rho} = -\frac{1}{2}\widetilde{\Box}_g\,\gamma_{\mu\rho} + O(\partial\gamma)^2$$

Moreover we have

$$g_{\mu\rho}R = g_{\mu\rho}g^{\alpha\beta}R_{\alpha\beta} = -\frac{1}{2}\eta_{\mu\rho}\widetilde{\Box}_g(\eta^{\alpha\beta}\gamma_{\alpha\beta}) + O(\gamma\partial^2\gamma) + O(\partial\gamma)^2$$

and hence with $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$:

$$G_{\mu\rho} = -\frac{1}{2}\Box\,\overline{\gamma}_{\mu\rho} + O(\gamma\partial^2\gamma) + O(\partial\gamma)^2, \qquad \text{where} \quad \overline{\gamma}_{\mu\rho} = \gamma_{\mu\rho} - \frac{1}{2}\eta_{\mu\rho}\eta^{\alpha\beta}\gamma_{\alpha\beta}.$$