Lecture 12: 5.1-2 Homogeneous and isotropic cosmology - the big bang. We will now make the assumption that space is *homogeneous* and *isotropic*. This is a reasonable assumption at the scale of the whole universe, where galaxies are viewed as particles. Homogeneous means that at any given instant of time each point of space should look like any other point. Isotropic means that at each time it looks the same in every space direction at each point. More precisely, we first assume that there is a foliation of spacetime into hypersurfaces Σ_t such that t is the proper time along the flow of the normal vectors to Σ_t and hence the spacetime metric can be written:

$$g_{ab} = -dt^2 + h_{ab}(t,x)dx^a dx^b$$

where $h_{ab}(t)$ is the restriction of the metric to g_{ab} to Σ_t. Homogeneous then means that for each t and any points $p, q \in \Sigma_t$ there is an isometry of the spacetime that also is an isometry of Σ_t that takes p to q. Isotropy means that for any given point $p \in \Sigma_t$ and two tangent vectors s_1 and s_2 at p tangential to Σ_t there is an isometry of spacetime that also is an isometry of Σ_t that leaves p and the normal at p fixed and takes s_1 to s_2.

Consider the Riemann curvature tensor ${}^{(3)}R_{abc}{}^d$ constructed from h_{ab} on Σ_t. We may view ${}^{(3)}R_{ab}{}^{cd}$ at a point p as a linear map, L, of the vector space W of two-forms (i.e. antisymmetric $(0,2)$ tensors) at p into itself $L : W \to W$. By the symmetries of the curvature tensor L is symmetric so it has an orthonormal basis of eigenvectors and by isotropy all eigenvalues must be the same and hence L must be a multiple of the identity $L = KI$ and hence

$${}^{(3)}R_{ab}{}^{cd} = K\delta^c{}_{[a}\delta^d{}_{b]}$$

By homogeneity K must be constant. It follows that the scalar curvature is constant and the spaces with constant scalar curvature are known. If $K > 0$ it is the 3 dimensional spheres, defined as a subset of points $\mathbf{R}^4$ of distance R to the origin. If $K = 0$ it is flat Euclidean space. If $K < 0$ is the 3 dimensional hyperboloid, defined as the subset of Minkowski space satisfying $t^2 - x^2 - y^2 - z^2 = R^2$. The spacetime metric can therefore be of either of the form in spherical, rectangular respectively hyperbolic coordinates

$$d^2 = -d\tau^2 + a(\tau)^2 \begin{cases} d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\,d\phi^2), & \text{if } k = 1 \\ dx^2 + dy^2 + dz^2, & \text{if } k = 0 \\ d\psi^2 + \sinh^2\psi\,(d\theta^2 + \sin^2\theta\,d\phi^2), & \text{if } k = -1 \end{cases}$$

These are called Robertson-Walker spacetimes.

Our aim is to substitute these spacetimes into Einstein's equations to solve for $a(\tau)$. We will have to cosmological model of matter. On a cosmic scale each galaxy can be though of as a grain of dust. The dust is modeled by the perfect fluid matter with vanishing pressure $P = 0$. However a thermal radiation fills the universe with pressure $P = \rho/3$. Although its contribution now is negligible this radiation is predicted to have made a dominant contribution in the early stages of the universe. We will therefore assume that

$$T_{ab} = \rho\, u_a u_b + P(g_{ab} + u_a u_b)$$

where u_a is the unit tangent to the world line of a the particles. If we plug this and the metric above into Einstein's equations we get that $a(\tau)$ must satisfy

$$3\dot{a}^2/a^2 = 8\pi\,\rho - 3k/a^2$$
$$3\ddot{a}/a = -4(\rho + 3P)$$

Several interesting facts can be read off from these formulas. First given that the universe is expanding $\dot{a} > 0$ it follows from the second equation and the fact that $\rho, P \geq 0$ that $\ddot{a} < 0$ and hence the universe must have expanded even more in the earlier stages so at some point in the past it must have had zero volume $a = 0$ and infinite density and curvature. This is what is called the *big bang*.

Before discussing the future evolution of the universe we need to get an equation for the evolution of the density of the universe. Multiplying the first equation by a^2, differentiating with respect to t and eliminating $\ddot{a}$ using the second equation gives

$$\dot{\rho} + 3(\rho + P)\dot{a}/a = 0$$

Thus for dust ($P = 0$)we find that ρa^3 is constant, whereas for radiation ($P = \rho/3$) we find that ρa^4 is constant. In either case we conclude from the first equation above that if $k = -1$ or $k = 0$ then $\dot{a}$ can never become zero so the universe expand forever. If $k = 1$ eventually the second term will dominate and $\dot{a} < 0$ and since $\ddot{a} < 0$ it will continue to decrease and after some finite time $a = 0$. This is what is called the *big crunch*.

5.3bParticle Horizons. The following question arises in the study of cosmological models in general relativity. In principle how much of our universe can be observed at a given event P. More precisely, which observers could have send a signal which reaches a given observer at event P. The boundary between the world lines that could be seen at P and those that can not is called the *particle horizon* at P. One might expect that all observers can communicate with each other by sending signals to each other sufficiently far back in time. However, this turn out not to be the case for Robinson-Walker spacetimes that expand rapidly. In fact suppose we have a metric

$$ds^2 = -d\tau^2 + a(\tau)^2(dx^2 + dy^2 + dz^2)$$

and make the change of time coordinate

$$t = \int \frac{d\tau}{a(\tau)}$$

so the metric becomes

$$ds^2 = a(t)^2(-dt^2 + dx^2 + dy^2 + dz^2).$$

In these coordinates it is possible to joint two events by a timelike curve if and only if it is possible to join them by a timelike curve in the Minkowski metric, the case $a = 1$. If the integral above converges then the spacetime only extends backwards to some finite time and it is hence impossible for an observer to receive signals from all of space.