Lecture 14: 6.3 Geodesics in the Schwarzschild spacetime.

Recall the expression for the *Schwarzschild metric*:

$$ds^2 = -\Big(1-\frac{2M}{r}\Big)dt^2 + \Big(1-\frac{2M}{r}\Big)^{-1}dr^2 + r^2\big(d\theta^2+\sin^2\theta\,d\phi^2\big)$$

It is the relativistic version of Newton's gravitational potential $\phi = CM/r$ that satisfy Poisson's equation $\triangle\phi = 0$ outside a body of mass M. Similarly the Schwarzschild metric satisfy Einstein's equations outside the body.

We will now study paths of particles in this field. Particles travel along geodesics. Massive particles along timelike geodesics and massless particles like light along null geodesics. If $u = dx/d\tau$ is the tangent to the geodesic then we can parameterize the curve so

$$g_{ab}u^a u^b = -\kappa$$

where $\kappa = 1$ for a massive particle and 0 for a massless particle. The geodesic equations are

$$\dot{u}^a + \Gamma^a_{bc}u^b u^c = 0$$

We want to solve this system of ordinary differential equations expressed in the $x = (t, r, \theta, \phi)$ coordinates. Then $u = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$. However there are quite a few equations but there are some quantities that are conserved along a geodesic. Apart from that the length of the tangent vector is constant the inner product between any killing vector field ξ and the tangent is constant. Moreover we may assume that the particle travels in the equatorial plane $\theta = \pi/2$ since it will travel in some plane determine by its velocity vector and the acceleration vector towards the origin so we can rotate the coordinate system so it becomes the equatorial plane.

A vector field ξ is called *killing* if

$$\mathcal{L}_\xi g_{ab} = 0 \qquad \Leftrightarrow \qquad \nabla_a\xi_b + \nabla_b\xi_a = 0$$

Here the Lie derivative is

$$\mathcal{L}_\xi g_{ab} = \xi^c\partial_c g_{ab} - (\partial_a\xi^c)g_{cb} - (\partial_b\xi^c)g_{ac}$$

If ξ is killing then

$$\frac{d}{d\tau}\big(g_{ab}\xi^a u^b\big) = u^b\nabla_b(\xi_a u^a) + u^b(\nabla_b\xi_a)u^a + \xi_a u^b\nabla_b u^a = 0$$

where the first term vanishes since ξ is killing and the second term since u is the tangent to a geodesic curve. The killing vector fields are the generators of the invariances. Since the Schwarzschild metric does not depend on t and ϕ it follows that $\mathcal{L}_\xi g_{ab} = 0$ if $\xi = \partial_t$ and $\mathcal{L}_\eta g_{ab} = 0$, if $\eta = \partial_\phi$. Hence we conclude that

$$E = g_{ab}u^a\xi^b = (1-2M/r)\dot{t}$$

is constant and so is

$$L = g_{ab}u^a\eta^b = r^2\dot{\phi}$$

We can now substitute these equations into the equations saying that the length of the tangent vector is constant as well

$$-\kappa = -(1 - 2M/r)\dot{t}^2 + (1 - 2M/r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2$$

to obtain

$$\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2, \qquad \text{where} \qquad V(r) = \frac{1}{2}\kappa - \kappa\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

This is now just one ordinary differential equation that can be solved for r and once r is know the others are easy to find. Note the similarity with the Newtonian potential. By differentiating the above equation we also get the second order equation

$$\ddot{r} = -V'(r)$$

If we view this as a first order system for $(r, \dot{r})$ the fixed points are the critical points where the right hand side vanishes. If $\kappa = 1$:

$$V'(r) = r^{-4}\big(Mr^2 - L^2 r + 3ML^2\big) = 0, \qquad \dot{r} = 0$$

The equation has roots

$$R_{\pm} = \frac{L^2}{2M}\Big(1 \pm \sqrt{L^2 - 12M^2}\Big)$$

Thus if $L^2 < 12M^2$ there are no extrema but if $L^2 > 12M^2$ there are two extrema

$$3M < R_- < 6M < R_+$$

V has a maximum at R_- and a minimum at R_+. The minimum R_+ is a stable equilibrium and the maximum R_- is an unstable equilibrium. This can been seen from looking at the eigenvalues of the linearized system around the equilibrium point or just from looking at the equation

$$\ddot{r} = -V'(r) \sim -V''(R_{\pm})(r - R_{\pm})$$

at the equilibrium point. Since $V''(R_-) < 0$ the particle will be pushed away; if $r > R_-$ then it will accelerate more to positive r and if $r < R_-$ it will accelerate to smaller r. If $\kappa = 0$ the situation is even simpler and the only root is $r = 3M$.