Lecture 2: 2.1 Manifolds.

A *topological manifold* M of dimension n is a Hausdorff topological space with a countable basis of open sets that locally looks like $\mathbf{R}^n$ in the following sense each $p \in M$ has a neighborhood which is homeomorphic to an open subset of $\mathbf{R}^n$. A homeomorphism is a continuous invertible function with continuous inverse.

The pair (U, ϕ) where O is an open subset of M and $\phi : O \to U$ is a homeomorphism onto an open subset U of $\mathbf{R}^n$ is called a coordinate neighborhood.

A C^∞ *manifold* M of dimension n is a topological manifold with a family of coordinate charts $\{(O_\alpha, \phi_\alpha)\}$ such that
(i) $\cup_\alpha O_\alpha = M$.
(ii) If $O_\alpha \cap O_\beta \neq \emptyset$ then $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(O_\alpha \cap O_\beta) \to \phi_\beta(O_\alpha \cap O_\beta) \to$ is a C^∞ map.

Ex $\mathbf{R}^n$ is a manifold, in fact with $O = \mathbf{R}^n$ and the identity map.
Ex The sphere $S^2 = \{(x^1, x^2, x^3) \in \mathbf{R}^3;\ (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$ is a manifold. In fact, the coordinate charts are the open sets $O_i^\pm = \{(x^1, x^2, x^3) \in S^2;\ \pm x^i > 0\}$ and the maps are the projections to the unit discs in the coordinate planes.

Let M and M' be C^∞ manifolds with charts $\{(O_\alpha, \phi_\alpha)\}$ and $\{(O'_\alpha, \phi'_\alpha)\}$ respectively.

Then the product $M \times M'$ is a C^∞ manifold with charts $\{(O_\alpha \times O'_\beta, (\phi_\alpha, \phi'_\beta))\}$.

We say that a map $f : M \to M'$ is C^∞ if $\phi'_\beta \circ f \circ \phi_\alpha^{-1} : U_\alpha \to U'_\beta$ is C^∞ for any α and β. If f is also one-to-one and onto with C^∞ inverse we say that f is a diffeomorphism, and that M and M' are diffeomorphic.

2.2 Tangent vectors.

Euclidean space has a natural vector space structure, we can e.g. add forces to get the total force acting on a particle. There is no such addition on a manifold. Instead we hope to be able to add infinitesimal displacement at a point.
For a manifold embedded in Euclidean space there is a natural notion of tangent vectors and tangent space. However, even though all manifolds can be embedded in Euclidean space there might not be a natural embedding so we would like to have a more intrinsic definition. We will use that there is a one to one correspondence between vectors and directional derivatives in $\mathbf{R}^n$: $(v^1, \dots, v^n) \to v^\mu \partial_\mu$.

We define a tangent vector at a point $p \in M$ in a manifold M to be a map $C^\infty(M, \mathbf{R}) \to \mathbf{R}$ which is linear and satisfies Leibnitz rule:
(i) $v(af + bg) = av(f) + bv(g)$
(ii) $v(fg) = f(p)v(g) + g(p)v(f)$
It is clear from the definition that the space of tangent vectors at a point V_p is a vector space. In fact its an n dimensional vector space:
Th If M is an n dimensional smooth manifold and $p \in M$ then $\dim V_p = n$.
In fact let (U, ϕ) be a chart containing p and define $X_\mu(f) = \dfrac{\partial}{\partial x^\mu}(f \circ \phi^{-1})\Big|_{\phi(p)}$, where $(x^1, \dots, x^n)$ are the coordinates in $\phi(O) \subset \mathbf{R}^n$.
Then X_μ, $\mu = 1, \dots, n$, are linearly independent since $X_\mu(x^\nu \circ \phi) = \delta_\mu^\nu$.
Using the Taylor expansion of f one can show that $v(f) = v^\mu X_\mu(f)$, $v^\mu = v(x^\mu \circ \phi)$.

Suppose that we have a manifold M and we have two different systems of coordinates. Then we have two possible expressions for a tangent vector:

$$v = v^\mu \frac{\partial}{\partial x^\mu} = v'^\nu \frac{\partial}{\partial x'^\nu}$$

The change of variable going from $x \to x'$ gives

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu}$$

Substituting this gives

$$v'^\nu = v^\mu \frac{\partial x'^\nu}{\partial x^\mu}$$

There is an alternative characterization of tangent space at a point $p \in M$ as an equivalence class of smooth curves going through p that have the same tangent vector at p in any coordinate system. A smooth curve through $p \in M$ is a smooth map from an interval $I \ni 0$ to the manifold $C : I \to M$, such that $C(0) = p$. The curve defines a tangent vector by for any $f \in C^\infty(M, \mathbf{R})$ in local coordinates $x(t) = \phi \circ C(t)$;

$$T(f) = \frac{d}{dt} f \circ C(t)\Big|_{t=0} = \frac{dx^\mu}{dt}\Big|_{t=0} \frac{\partial}{\partial x^\mu}\left(f \circ \phi^{-1}\right)$$

We have so far defined the space of tangent vector at a point V_p. A smooth vector field is an assignment of an tangent vector $v\big|_p \in V_p$ at each point $p \in M$, that vary smoothly with p in the sense that $v(f)$ is a smooth function of p for each smooth f. This is equivalent to that the component functions are v^μ are smooth.

A *one-parameter group of diffeomorphisms* ϕ_t is a map $I \times M \to M$ such that $\phi_t : M \to M$ is a diffeomorphism, for each $t \in I$ and $\phi_{t+s} = \phi_t \circ \phi_s$. The tangent vector to the curve $t \to \phi_t$ at $t = 0$ is a vector field. Conversely, given a vector field its integral curves are solutions of the system of ordinary differential equations in local coordinates

$$\frac{dx^\mu}{dt} = v^\mu(x^1, \ldots, x^n)$$

2.3 Tensors and the cotangent space. An example of tensor in physics is the stress tensor defined as follows. Consider a small plane segment with unit normal $\mathbf{n}$. Let $\mathbf{F}$ be the force that a small mass on one side of the plane asserts on an equal mass on the other side of the plane and let ℓ be another unit vector. The stress tensor $T(\mathbf{n}, \ell) = T_{ij} n^i \ell^j$ is the component of the force $\mathbf{F}$ in the ℓ direction.

Let V be a vector space of dimension n and let V^* be the *dual vectors space*, of linear maps $f : V \to \mathbf{R}$. If $v_1, \ldots, v_n$ is a basis for V the we can find a dual basis $v^{*1}, \ldots, v^{*n}$ for V^* such that $v^{*\mu}(v_\nu) = \delta^\mu_\nu$. (Here $\delta^\nu_\mu = 1$, if $\mu = \nu$ and $\delta^\mu_\nu = 0$ if $\mu \neq \nu$.) The correspondence $v_\mu \to v^{*\mu}$ gives an isomorphism between V and V^*, but this isomorphism depends on the choice of basis $\{v_\mu\}$ so there is no natural way of identifying V and V^* unless we have a metric. The dual of the dual V^{**} can however naturally be identified with V. In fact if $v \in V$ and $w^* \in V^*$ then $v(w^*) = w^*(v)$ defines an element of V^{**} and this map must be onto since the dimension of V is the same as that of V^{**}.