Lecture 3: 2.3 Tensors and the cotangent space. An example of tensor in physics is the stress tensor defined as follows. Consider a small plane segment with unit normal $\mathbf{n}$. Let $\mathbf{F}$ be the force that a small mass on one side of the plane asserts on an equal mass on the other side of the plane and let ℓ be another unit vector. The stress tensor $T(\mathbf{n},\ell)=T_{ij}n^i\ell^j$ is the component of the force $\mathbf{F}$ in the ℓ direction.

Let V be a vector space of dimension n and let V^* be the *dual vectors space*, of linear maps $f:V\to\mathbf{R}$. If $v_1,\dots,v_n$ is a basis for V the we can find a dual basis $v^{*1},\dots,v^{*n}$ for V^* such that $v^{*\mu}(v_\nu)=\delta^\mu_\nu$. (Here $\delta^\nu_\mu=1$, if $\mu=\nu$ and $\delta^\mu_\nu=0$ if $\mu\neq\nu$.) The correspondence $v_\mu\to v^{*\mu}$ gives an isomorphism between V and V^*, but this isomorphism depends on the choice of basis $\{v_\mu\}$ so there is no natural way of identifying V and V^* unless we have a metric. The dual of the dual V^{**} can however naturally be identified with V. In fact if $v\in V$ and $w^*\in V^*$ then $v(w^*)=w^*(v)$ defines an element of V^{**} and this map must be onto since the dimension of V is the same as that of V^{**}.

A *tensor, T, of type (k,ℓ) over V*, is a multilinear map of k copies of V^* and ℓ copies of V to $\mathbf{R}$:

$$T:V^*\times\cdots\times V^*\times V\times\cdots\times V\to\mathbf{R}$$

A contraction of a tensor T is the tensor $CT=T(\dots,v^{*\mu},\dots;\dots,v_\mu,\dots)$. This can be seen to be independent of basis just like the trace of a matrix is.

Given a tensor T of type (k,ℓ) and a tensor T' of type (k',ℓ') we can construct a new tensor of type $(k+k',\ell+\ell')$ called the outer product $T\otimes T'$ by

$$\begin{aligned}T\otimes T'\big(&v^{*1},\dots,v^{*k+k'},v_1,\dots,v_{\ell+\ell'}\big)\\&=T\big(v^{*1},\dots,v^{*k},v_1,\dots,v_\ell\big)\;T\big(v^{*k+1},\dots,v^{*k+k'},v_{\ell+1},\dots,v_{\ell+\ell'}\big)\end{aligned}$$

Given a basis $v_1,\dots,v_n$ of V and a dual basis $v^{*1},\dots v^{*n}$ of V^* we can express a general tensor T of type (k,ℓ) as

$$T=T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_\ell}\;v_{\mu_1}\otimes\cdots\otimes v_{\mu_k}\otimes v^{*\nu_1}\otimes\cdots\otimes v^{*\nu_\ell}.$$

We are in particular interested in the case $V=V_p$ is the tangent space at a point p of a manifold M. In that case we can choose the basis for V to be the vectors $\partial/\partial x^1,\dots,\partial/\partial x^n$ in some local coordinates. It is then customary to denote the dual basis by $dx^1,\dots,dx^n$, i.e. dx^μ is the linear transformation on the tangent space such that $dx^\mu(\partial/\partial x^\nu)=\delta^\mu_\nu$. There are also a couple of other ways to interpret the notation dx^μ. If $f:M\to\mathbf{R}$, the differential df at p is the linear map $V_p\to\mathbf{R}_{f(p)}=\mathbf{R}$ of tangent vectors given by $df(v)=v(f)$. Hence if dx^μ denotes the differential of the coordinate function x^μ we have $dx^\mu(\partial/\partial x^\nu)=\partial/\partial x^\nu(x^\mu)=\delta^\mu_\nu$.

There is however, an alternative way of describing the cotangent space V_p^* as equivalence classes of functions $f:M\to\mathbf{R}$ vanishing at p modulo functions vanishing to higher order or products of functions vanishing at p. Then df denotes the equivalence class of f. This is completely analogous to the description of a tangent vector $v=C'(t)$ as an equivalence classes of curves $C:\mathbf{R}\to M$, that have the same tangent vector at $t=0$ in any local coordinates. It is clear that the pairing $df(v)=v(f)$ is independent of representative. One can think of cotangent vectors as level sets of functions and the corresponding tangent vector as a normal to the level set.

The reason we need to consider both tangent and cotangent vectors is that both show up naturally. The tangent vectors are the derivatives of functions to the manifolds such as the tangent vector to a path of a particle and cotangent vectors are the derivatives of functions on the manifold such as the differential of the temperature. The paring between the velocity vector to the particle and the temperature differential is then the change of temperature the particle feels.

The reason we need to differentiate between tangent and cotangent vectors is that they transform differently under changes of coordinates. Using the pairing we see that the components of a cotangent vector transform like

$$\omega'_{\mu'} = \omega_\mu \frac{\partial x^\mu}{\partial x^{\mu'}}$$

Its easy to check that the components of a tensor T of type (k,ℓ) transform like

$$T'^{\mu_1'\cdots\mu_k'}{}_{\nu_1'\cdots\nu_\ell'} = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} \frac{\partial x'^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\mu_k'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x'^{\mu_1'}} \cdots \frac{\partial x^{\nu_\ell}}{\partial x'^{\mu_\ell'}}$$

One can alternatively define a tensor to be something that transform in this way.

An assignment of a tensor over V_p for each $p \in M$ is called a *smooth tensor field* if the component functions are smooth functions of p or equivalently if the tensor applied to smooth vector fields and covector fields are smooth functions.

The metric g is a symmetric non-degenerate $(0,2)$ tensor field, i.e. a quadratic form on the tangent space, if X, Y are tangent vector then $g(X,Y)$ is a real number, such that $g(X,Y) = g(Y,X)$ and $g(X,X) = 0$ if and only if $X = 0$. If we express $X = X^\mu \partial_\mu$, $Y = Y^\nu \partial_\nu$ in a basis we have

$$g(X,Y) = g_{\mu\nu} X^\mu Y^\nu,$$

We can write this as

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$$

We will sometime write this as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

We can always find an orthonormal basis so that $g(v_i, v_j) = 0$, if $i \neq j$ and $g(v_i, v_i) = \pm 1$. The metric is called Riemannian if the signature are $+,+,+,+$ and Lorenzian if it is $-,+,+,+$.

The metric induces a map from the tangent to the cotangent space by $\mathbf{X} \to g(\,\cdot\,, \mathbf{X})$.

2.4 Abstract index notation. The *abstract index notation* is almost the same as writing the tensor in components in an arbitrary basis without haven chosen a particular basis. The idea is not to introduce a basis but use a notation for tensors that mirrors the component notation. In this notation a tensor of type (k,ℓ) will be denoted by $T^{a_1\cdots a_k}{}_{b_1\cdots b_\ell}$, where we use latin letters. E.g. the expression $T_{abc}X^aY^bZ^c$ simply means $T(X,Y,Z)$, i.e. a,b,c is not supposed to be summed over the components in a basis but are just labels standing for the first, second and third argument. The difference is not one of substance since $T(X,Y,Z)$ is equal to the sum over the components $T_{\alpha\beta\gamma}X^\alpha Y^\beta Z^\gamma$ in any basis.

We define the symmetrization of a $(0,2)$ tensor to be

$$T_{(ab)} = \frac{1}{2}\big(T_{ab} + T_{ba}\big)$$

and the anti-symmetrization to be

$$T_{[ab]} = \frac{1}{2}\big(T_{ab} - T_{ba}\big)$$

Similarly in higher dimensions. A totally anti-symmetric $(0,\ell)$ tensor

$$T_{a_1\ldots a_\ell} = T_{[a_1\ldots a_\ell]}$$

is called a differential ℓ form.