

Lecture 4: 2.4 Abstract index notation. The *abstract index notation* is almost the same as writing the tensor in components in an arbitrary basis without having chosen a particular basis. The idea is not to introduce a basis but use a notation for tensors that mirrors the component notation. In this notation a tensor of type (k, ℓ) will be denoted by $T^{a_1 \dots a_k}_{b_1 \dots b_\ell}$, where we use latin letters. E.g. the expression $T_{abc} X^a Y^b Z^c$ simply means $T(X, Y, Z)$, i.e. a, b, c is not supposed to be summed over the components in a basis but are just labels standing for the first, second and third argument. The difference is not one of substance since $T(X, Y, Z)$ is equal to the sum over the components $T_{\alpha\beta\gamma} X^\alpha Y^\beta Z^\gamma$ in any basis.

We define the symmetrization of a $(0, 2)$ tensor to be

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba})$$

and the anti-symmetrization to be

$$T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba})$$

Similarly in higher dimensions. A totally anti-symmetric $(0, \ell)$ tensor

$$T_{a_1 \dots a_\ell} = T_{[a_1 \dots a_\ell]}$$

is called a differential ℓ form.

3.1 Covariant Derivative and Parallel transport.

The covariant derivative of a (k, ℓ) tensor field T is a $(k, \ell + 1)$ tensor field ∇T satisfying the rules below. With some abuse of the index notation we will write $\nabla_c T^{a_1 \dots a_k}_{b_1 \dots b_\ell}$ for the tensor $(\nabla T)^{a_1 \dots a_k}_{c b_1 \dots b_\ell}$

1. Linearity: $\nabla_c (\alpha A^{a_1 \dots a_k}_{b_1 \dots b_\ell} + \beta B^{a_1 \dots a_k}_{b_1 \dots b_\ell}) = \alpha \nabla_c A^{a_1 \dots a_k}_{b_1 \dots b_\ell} + \beta \nabla_c B^{a_1 \dots a_k}_{b_1 \dots b_\ell}$

2. Leibnitz rule:

$$\nabla_c (A^{a_1 \dots a_k}_{b_1 \dots b_\ell} B^{c_1 \dots c_k}_{d_1 \dots d_\ell}) = (\nabla_c A^{a_1 \dots a_k}_{b_1 \dots b_\ell}) B^{c_1 \dots c_k}_{d_1 \dots d_\ell} + A^{a_1 \dots a_k}_{b_1 \dots b_\ell} \nabla_c B^{c_1 \dots c_k}_{d_1 \dots d_\ell}$$

3. Commutativity with contractions $\nabla_c (A^{a_1 \dots d \dots a_k}_{b_1 \dots d \dots b_\ell}) = \nabla_c A^{a_1 \dots d \dots a_k}_{b_1 \dots d \dots b_\ell}$.

4. Consistence with notion of tangent vector:

If $t = t^\mu \partial_\mu$ is a vector field and f a function then $t(f) = t^a \nabla_a f$.

5. Torsion free: If f is a function $\nabla_a \nabla_b f = \nabla_b \nabla_a f$.

In a fixed coordinate system the ordinary derivative $\partial_a = \partial/\partial x^a$ satisfy the rules. However, this is not invariant under changes of coordinates. In fact, if $X = X^a \partial/\partial x^a = X'^a \partial/\partial x'^a$ is a vector field expressed in different coordinates then

$$\frac{\partial X'^b}{\partial x'^a} = \frac{\partial x^c}{\partial x'^a} \frac{\partial}{\partial x^c} \left(\frac{\partial x'^b}{\partial x^d} X^d \right) = \frac{\partial x^c}{\partial x'^a} \frac{\partial x'^b}{\partial x^d} \frac{\partial X^d}{\partial x^c} + \frac{\partial x^c}{\partial x'^a} \left(\frac{\partial}{\partial x^c} \frac{\partial x'^b}{\partial x^d} \right) X^d.$$

In other words, if $\partial X^d/\partial x^c$ is the tensor in the x coordinates then the tensor expressed in the x' coordinates is different from $\partial X'^a/\partial x'^b$.

One can show that if $\tilde{\nabla}_a = \partial/\partial x^a$ then any covariant derivative must be of the form

$$\nabla_a t^b = \tilde{\nabla}_a t^b + C^b_{ac} t^c$$

A vector field v^a is said to be *parallel transported* along a curve with tangent t^a if

$$t^a \nabla_a v^b = 0$$

or if we use the above equations

$$t^a \partial_a v^b + t^a C^b_{ac} v^c = 0.$$

Recall that a metric g is a $(0, 2)$ tensorfield.

Th If g_{ab} is the metric there is a unique covariant derivative satisfying $\nabla_c g_{ab} = 0$.

The proof of this also gives that C^b_{ac} are just the Christoffel symbols Γ^b_{ac} given by

$$\Gamma^b_{ac} = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab})$$

(3.1.1)

$$\nabla_c T^{a_1 \dots a_k}_{b_1 \dots b_\ell} = \partial_c T^{a_1 \dots a_k}_{b_1 \dots b_\ell} + \sum_{i=1}^k \Gamma^{a_i}_{cd} T^{a_1 \dots d \dots a_k}_{b_1 \dots b_\ell} - \sum_{i=1}^{\ell} \Gamma^d_{cb_i} T^{a_1 \dots a_k}_{b_1 \dots d \dots b_\ell}$$