Lecture 5: 3.1 Covariant derivative. In a Riemannian manifold with metric g one can define a derivative operator on tensors by
(3.1.1)

$$\nabla_c T^{a_1\cdots a_k}{}_{b_1\cdots b_\ell} = \partial_c T^{a_1\cdots a_k}{}_{b_1\cdots b_\ell} + \sum_{i=1}^{k} \Gamma^{a_i}_{cd}\, T^{a_1\cdots d\cdots a_k}{}_{b_1\cdots b_\ell} - \sum_{i=1}^{\ell} \Gamma^{d}_{cb_i}\, T^{a_1\cdots a_k}{}_{b_1\cdots d\cdots b_\ell}$$

where the Christoffel symbols Γ^b_{ac} are given by

$$\Gamma^b_{ac} = \frac{1}{2} g^{cd}\big(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\big)$$

3.2 Curvature.

A calculation shows that

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)(f\omega_c) = f(\nabla_a\nabla_b - \nabla_b\nabla_a)\,\omega_c$$

It follows that this quantity at a point in the manifold only depends on ω_c at that point and not on the derivatives of ω_c. Hence there is four tensor $R_{abc}{}^d$ such that

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\,\omega_c = R_{abc}{}^d \omega_d. \tag{3.2.1}$$

$R_{abc}{}^d$ is called the *Riemann curvature tensor.* A calculations shows that

$$0 = (\nabla_a\nabla_b - \nabla_b\nabla_a)(t^c\omega_c) = \omega_c(\nabla_a\nabla_b - \nabla_b\nabla_a)\,t^c + t^c(\nabla_a\nabla_b - \nabla_b\nabla_a)\,\omega_c$$

and hence

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\,t^c = -R_{abd}{}^c\, t^d. \tag{3.2.2}$$

By induction
(3.2.3)

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)T^{c_1\cdots c_k}{}_{d_1\cdots d_\ell} = \sum_{i=1}^{\ell} R_{abd_j}{}^{e}\, T^{c_1\cdots c_k}{}_{d_1\cdots e\cdots d_\ell} - \sum_{i=1}^{k} R_{abe}{}^{e_i}\, T^{c_1\cdots e\cdots c_k}{}_{d_1\cdots d_\ell}$$

The Riemann curvature tensor satisfy the following properties

1. $R_{abc}{}^d = -R_{bac}{}^d$. This follows directly from (3.2.1).

2. $R_{[abc]}{}^d = 0$. Here $[abc]$ is stands for the antisymmetrization over the indices a, b, c: $R_{[abc]}{}^d = R_{abc}{}^d + R_{bca}{}^d + R_{cab}{}^d - R_{bac}{}^d - R_{bca}{}^d - R_{acb}{}^d$ so if we also use (1) we get: $R_{abc}{}^d + R_{bca}{}^d + R_{cab}{}^d = 0$.

3. $R_{abcd} = -R_{abdc}$, if $R_{abcd} = R_{abc}{}^e g_{de}$. This follows from (3.2.2) since $\nabla_a g_{bc} = 0$. Moreover (1)-(3) in addition implies $R_{abcd} = R_{cdab}$.

4. The Bianchi identity $\nabla_{[a} R_{bc]d}{}^e = 0$.

Using (3.1.1) and $\Gamma^c_{ab} = \Gamma^c_{ba}$ one can show that $\nabla_{[a}\nabla_b\omega_{c]} = 0$, from which (2) follows. (4) follows from using (3.2.14) and (2).