Lecture 7: 3.4a How to calculate the curvature.
Recall that the curvature was defined by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\,\omega_c = R_{abc}{}^d \omega_d.$$

We will now derive a formula for the curvature in terms of the metric. We have

$$\nabla_b \omega_c = \partial_b \omega_c - \Gamma_{bc}^d \omega_d$$

and

$$\nabla_a \nabla_b \omega_c = \partial_a \big(\partial_b \omega_c - \Gamma_{bc}^d \omega_d\big) - \Gamma_{ab}^e \big(\partial_e \omega_c - \Gamma_{ec}^d \omega_d\big) - \Gamma_{ac}^e \big(\partial_b \omega_e - \Gamma_{be}^d \omega_d\big)$$

using the symmetry $\Gamma_{ab}^d = \Gamma_{ba}^d$ and the commutativity of partial derivatives we get

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\,\omega_c = -\partial_a \Gamma_{bc}^d \omega_d + \partial_b \Gamma_{ac}^d \omega_d + \Gamma_{ac}^e \Gamma_{be}^d \omega_d - \Gamma_{bc}^e \Gamma_{ae}^d \omega_d$$

and hence

$$R_{abc}{}^d = -\partial_a \Gamma_{bc}^d + \partial_b \Gamma_{ac}^d + \Gamma_{ac}^e \Gamma_{be}^d - \Gamma_{bc}^e \Gamma_{ae}^d$$

Using that $\Gamma_{ab}^d = g^{dc}\big(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}\big)/2$ we see that $R_{abc}{}^d \sim \partial^2 g + (\partial g)^2$.

3.4b. We also did the method of frames as in the book.