Lecture 8: 4.1 The geometry of Space in Prerelativity Physics; Covariance.

In classical physics the metric of space is given by

$$ds^2 = dx^2 + dy^2 + dz^2$$

or in index notation

$$h_{\mu\nu} dx^\mu \, dx^\nu$$

where $h = \mathrm{diag}(1, 1, 1)$.

The *principle of covariance* states that all measurable quantities are obtained by contracting tensors with basis vectors and that the resulting scalars should have the same value in all frames. Moreover the physical equations should only involve tensors, its derivatives and the metric.

4.2 Special Relativity.

Special relativity is similar with the exception that the metric is replaced by a space-time metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

or in index notation

$$\eta_{\mu\nu} dx^\mu \, dx^\nu$$

where $\eta = \mathrm{diag}(-1, 1, 1, 1)$. The principle of covariance still holds but with respect to the space-time metric. They should be invariant under Lorentz transformation as well as rotations and translations of space.

We have already mentioned that curves with tangent T in spacetime are classified as timelike, null or spacelike according to if $\eta_{ab} T^a T^b$ is negative, zero or positive. The difference in t coordinate between two events have no meaning since it depends on the choice of coordinate system. However we can define the *proper time* along a timelike curve to be

$$\tau = \int \sqrt{-\eta_{ab} T^a T^b} \, dt.$$

where T is the unit tangent vector to the curve. This is invariant under changes of parameter along the curve. However, two curves with the same start and end points might have different proper time which leads to the so called twin paradox. The tangent vector u^a to a timelike curve parameterized by proper time has unit length:

$$u^a u_a = -1.$$

A particle subject to no external forces will travel along a geodesic:

$$\frac{d}{d\tau} u^b = u^a \partial_a u^b = u^a \nabla_a u^b = 0$$

since the Christoffel symbols of the metric vanish.
All particles have a *rest mass* m. The *energy-momentum* 4-vector, p^a, of a particle of mass m is defined by

$$p^a = m u^a$$

The *energy* of a particle as measured by an observer-present at the site of the particle-whose 4-velocity is v^a is defined by

$$E = -p_a v^a.$$

For a particle at rest with respect to the observer (i.e. $v^a = u^a$) this reduces to the familiar formula $E = mc^2$ (in our units $c = 1$).

Continuous matter distribution is described by a symmetric tensor T_{ab} called the *stress-energy-momentum* tensor. For an observer with 4-velocity v^a, the components $T_{ab}v^a v^b$ is interpreted as the energy density, i.e. mass-energy per unit volume, as measure by the observer. For normal matter this is positive

$$T_{ab}v^a v^b \geq 0.$$

If x^a is orthogonal to v^a, then component $-T_{ab}v^a x^b$ is interpreted as the momentum density of matter in the x^a-direction. If y^a also is orthogonal to v^a, then $T_{ab}x^a y^b$ represents the x^a-y^b component of the stress tensor previously defined.

We will give examples of stress-energy tensors, for fluids and electromagnetism. The stress-energy tensor will depend on matter fields which satisfy certain equations. In particular we always have the divergence free condition

$$\partial^a T_{ab} = 0$$

In the non-relativistic limit of the matter field for a perfect fluid the divergence free condition leads to Euler's equations of a fluid. The divergence free condition in particular implies energy conservation.