Lecture 9: 4.2 The energy momentum tensor in special relativity.

The energy momentum tensor for a *perfect fluid* is given by

$$T_{ab} = \rho\, u_a u_b + P(\eta_{ab} + u_a u_b) = (\rho + P) u_a u_b + P\eta_{ab}$$

where u^a is a unit timelike vector field representing the 4-velocity of the fluid, ρ is the density, P the pressure and η_{ab} is the Minkowski metric. The equations of motion are

$$\partial^a T_{ab} = 0 \tag{4.2.0}$$

or

$$\big((\rho + P)\partial^a u_a + u^a \partial_a(\rho + P)\big) u_b + (\rho + P) u^a \partial_a u_b + \partial_b P = 0 \tag{4.2.1}$$

Since $u^b u_b = -1$ it follows from contracting with $-u^b$ that

$$(\rho + P)\partial^a u_a + u^a \partial_a \rho = 0 \tag{4.2.2}$$

The component of (4.2.1) along u_b is therefore (4.2.2) multiplied by u_b and subtracting it off yields the perpendicular component:

$$(u^a \partial_a P) u_b + (\rho + P) u^a \partial_a u_b + \partial_b P = 0. \tag{4.2.3}$$

In the nonrelativistic limit when $P << \rho$, $u^\mu \sim (1, V)$, and $|V| dP/dt << |\nabla P|$ these equations become

$$\partial_t \rho + \mathrm{div}(\rho V) = 0$$

and

$$\rho\big(\partial_t V_i + V^k \partial_k V_i\big) = -\partial_i P, \qquad i = 1, 2, 3.$$

The divergence free condition $\partial^a T_{ab}$ imply energy conservation. In fact, consider an observer with constant 4-velocity v^a, so that $\partial_b v^a = 0$. The quantity

$$J_a = -T_{ab} v^b$$

represents the mass-energy current density 4-vector of the fluid measured by these observers. Then

$$\partial^a J_a = -(\partial^a T_{ab}) v^b = 0$$

If S is the surface of a spacetime domain D then by the spacetime divergence theorem

$$\text{Flux of Energy out through } S = \int_S J_a n^a\, dS = \int_D \partial^a J_a\, dV = 0$$

Physically this means energy conservation.

The energy momentum tensor for a *scalar field* satisfying Klein-Gordon equation

$$\partial^a \partial_a \phi - m^2 \phi = 0 \tag{4.2.4}$$

(here $\partial^a \partial_a = -\partial_t^2 + \triangle_x$ is the wave operator) is

$$T_{ab} = \partial_a \phi \, \partial_b \phi - \frac{1}{2} \eta_{ab} \big(\partial^c \phi \, \partial_c \phi + m^2 \phi \big)$$

and $\partial^a T_{ab} = 0$ by (4.2.4).

In pre-relativity physics the electric field E and the magnetic field B are spacial vectors. In special relativity they can be combined to a single space-time tensor field F_{ab}, which is antisymmetric $F_{ab} = -F_{ba}$, and its dual $*F_{ab} = -\frac{1}{2} \epsilon_{ab}{}^{cd} F_{cd}$, where $\epsilon_{abcd} = 1$ if (a, b, c, d) is an even permutation of $(0, 1, 2, 3)$, its equal to $= -1$ if its an odd permutation and equal to 0 otherwise. Here as usual we have increased and lowered the indices with respect to the Minkowski metric η_{ab}. For an observer with 4-velocity v, $E_a = F_{ab} v^b$ is interpreted as the electric field and $B_a = *F_{ab} v^b$ the magnetic field.

In terms of F_{ab} the Maxwell equations take the simple form

$$\partial^a F_{ab} = -4\pi j_b$$

$$\partial_{[a} F_{bc]} = 0 = \partial^a * F_{ab}$$

where j^a is the current 4-vector of electric charge.

The energy momentum tensor for the *electromagnetic field* is

$$T_{ab} = \frac{1}{4\pi} \big(F_{ac} F_b{}^c - \frac{1}{4} \eta_{ab} F_{de} F^{de} \big)$$

In order for $\partial^a T_{ab} = 0$ we must have that $j^a = 0$ in order for Maxwell's equations above to hold. However, if we add the energy-momentum tensor for the scalar field to the one for the electromagnetic field the total energy momentum tensor can still be divergence free with $j^a \neq 0$. In this case we will get an additional term in the right hand side of the Klein Gordon equation.

A particle of charge q and mass m moving in the electromagnetic field F_{ab} will feel the acceleration

$$\frac{d}{d\tau} u^b = u^a \partial_a u^b = \frac{q}{m} F^b{}_c u^c$$

This is the special relativistic version of the Lorentz force law. The left is the acceleration measured in the particles own frame.