Math 250B Riemannian Geometry

Lecture 1: Overview. What is the course about?

Consider a submanifold of Euclidean space, e.g. a curve or a surface in space.
On the submanifold lives creatures that are unaware of the surrounding space.
They can measure distances on the submanifold but not distances in space.
How much of the geometry of the submanifold can they find out?
We can straighten out a curve or bend a paper and they wouldn't know the difference.

Gauss asked how much of the geometry of a surface is independent of how it bends in space, i.e. how much of the geometry that remains the same if we perform an isometry that doesn't change distances between points.
This is called the intrinsic geometry as apposed to the extrinsic geometry that measures how the submanifold lies in space.
Consider a curve in the plane. The curvature at a point is defined to be the inverse of the radius of the circle that is tangential to the curve to second order there.
Clearly the curvature of a curve is an extrinsic quantity.

Gauss Theorem Egregium:
The product of the principal curvatures of a surface is invariant under isometries.
At the point on a surface consider planes through the point containing the normal.
The intersection between the planes and the surface are curves.
As the angle of the plane changes so does the curvature of the curve.
The principal curvatures are the maximum curvature and the minimum curvature.

Riemann came up with a higher dimensional version of this theory which in addition can be formulated for an abstract Riemannian metric on the tangent space.
Consider a curve, with tangent vector $\dot{X}(t)$. Then the Riemannian distance is $\int \|\dot{X}(t)\|\, dt$, where $\|X'(t)\|^2 = g_{ij}\dot{X}^i\dot{X}^j$ and g_{ij} is called the Riemannian metric.
There is a curvature tensor R_{ijkl} which is invariant under isometries, i.e. can be calculated in terms of the metric.

Einstein realized that this theory could be used to describe how space curves under the influence of gravity which lead to the general theory of relativity.
These are equations for the curvature tensor.

Core topics:
Curvature of curves, surfaces and submanifolds of Euclidean space.
Riemannian metric on a manifold. Parallel transport and Riemannian connection.
Geodesics and the exponential map. Jacobi fields.
Curvature, Sectional curvature. Bianchi identities.
Submanifolds, immersions.
The second fundamental form. Codazzi equation.

Additional topics that we might cover:
General relativity.
Moving frames and the structural equations.

Curves in Euclidean space.

Consider a curve: $\mathbf{R} \supset I \ni t \to x(t) \in \mathbf{R}^N$, such that $\dot{x}(t) = \dfrac{dx}{dt}(t) \neq 0$, $t \in I$.
If $\phi : J \to I$ is a bijection we regard $x \circ \phi$ as the same curve.
The arc length $s(t)$ is defined up to an additive constant by $\dfrac{ds}{dt}(t) = \|\dot{x}(t)\|$.
If we define $X(s(t)) = x(t)$ we get a curve with $\|\dot{X}(s)\| = 1$.
Hence $\dfrac{d}{ds}\langle \dot{X}(s), \dot{X}(s)\rangle = 2\langle \ddot{X}(s), \dot{X}(s)\rangle = 0$, i.e. $\ddot{X}$ is perpendicular to $\dot{X}$.

Consider a circle with radius R parameterized by arc length:
$x(s) = x_0 + Re_1 \cos(s/R) + Re_2 \sin(s/R)$, where e_1 and e_2 are orthogonal unit vectors.
Then tangent is $\dot{x}(s) = -e_1 \sin(s/R) + e_2 \cos(s/R)$ and
the second derivative $\ddot{x}(s) = (-e_1/R)\cos(s/R) - (e_2/R)\sin(s/R) = (x_0 - x(s))/R^2$
is directed towards the center of the circle and $\|\ddot{x}\| = 1/R$.

Def If $x(s)$ is parametrized by arc length, i.e. $\|\dot{x}(s)\| = 1$ then $n = \ddot{x}(s)/\|\ddot{x}(s)\|$ is called the principal normal of the curve at $x(s)$ and $\kappa(s) = \|\ddot{x}(s)\|$ is the curvature. The plane spanned by $\dot{x}(s)$ and $n(s)$ is called the osculating plane.

The Frenet formulas.

Since the principal normal satisfies $\langle n(s), n(s)\rangle = 1$ we get $\langle \dot{n}(s), n(s)\rangle = 0$, and since $\langle n(s), \dot{x}(s)\rangle = 0$ we have $\langle \dot{n}(s), \dot{x}(s)\rangle + \langle n(s), \ddot{x}(s)\rangle = 0$ i.e.

$$\langle \dot{n}(s), \dot{x}(s)\rangle = -\kappa(s).$$

Hence $\dot{n}(s) + \kappa(s)\dot{x}(s)$ is orthogonal to the plane spanned by $\dot{x}(s)$ and $n(s)$. The length of this vector is called the torsion $\tau(s)$ and the normalized unit vector in its direction is called the binormal $b(s)$. We have

$$\dot{n}(s) = -\kappa(s)\dot{x}(s) + \tau(s)b(s)$$

Differentiation of the equations

$$\langle b(s), \dot{x}(s)\rangle = 0, \quad \langle b(s), n(s)\rangle = 0, \quad \langle b(s), b(s)\rangle = 1$$

gives

$$\langle \dot{b}(s), \dot{x}(s)\rangle = -\langle b(s), \ddot{x}(s)\rangle = 0, \quad \langle \dot{b}(s), n(s)\rangle = -\langle b(s), \dot{n}(s)\rangle = -\tau(s), \quad \langle \dot{b}(s), b(s)\rangle = 0$$

We obtain

$$\begin{aligned} \ddot{x} &= \qquad\qquad \kappa(s)n(s) \\ \dot{n}(s) &= -\kappa(s)\dot{x}(s) \qquad\qquad + \tau(s)b(s) \\ \dot{b}(s) &= \qquad\qquad -\tau(s)n(s) \end{aligned}$$

Given $\kappa(s)$ and $\tau(s)$ this system of differential equations has a unique solution for given initial conditions. The interpretation of this is as follows: the curvature $\kappa(s)$ measures how much the curve curves in its osculating plane and the torsion $\tau(s)$ measures how much the osculating plane turns as we go along the curve. These two quantities completely characterize the curve. In particular if it is a curve in the plane then the torsion is zero and the curve is completely determined by its curvature at each point and initial tangent.