Lecture 10: The Riemannian connection.

A connection is said to be **symmetric** if $\nabla_X Y - \nabla_Y X = [X,Y] = XY - YX$.
A connection is said to be **compatible** with a Riemannian metric $\langle\, ,\, \rangle$ if

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \qquad \text{where} \quad Xf = X^k \partial_k f$$

Theorem Given a Riemannian manifold, there is a unique connection that is symmetric and compatible with the metric. It is called the **Levi-Civita connection**.
Pf We have

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \tag{6}$$

$$Y\langle Z,X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \tag{7}$$

$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \tag{8}$$

Adding (6) and (7) and subtracting (8) and using the symmetry we get

$$X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle = \langle [X,Z],Y\rangle + \langle [Y,Z],X\rangle + \langle [X,Y],Z\rangle + 2\langle Z, \nabla_Y X\rangle$$

Therefore

$$\langle Z, \nabla_X Y\rangle = \frac{1}{2}\Big(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle - \langle [X,Y],Z\rangle \Big)$$

It follows that ∇ is uniquely determined from the metric.

In coordinates $\nabla_{E_i} E_j = \Gamma_{ij}^k E_k$, $E_i = \partial/\partial x^i$ we have $[E_i, E_j] = 0$ and the above becomes if $Z = E_k$, $X = E_i$ and $Y = E_j$

$$\Gamma_{ij}^\ell g_{\ell k} = \frac{1}{2}\Big(\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ki} - \frac{\partial}{\partial x^k} g_{ij} \Big)$$

With Γ_{ij}^k as above the formula for covariant derivative exactly the same as for a submanifold of $\mathbf{R}^N$, with the exception that g_{jk} now do not come from restricting the Euclidean metric but instead is abstractly given:

$$\frac{DV}{dt} = \Big(\frac{dV^k}{dt} + \Gamma_{ij}^k \frac{dx^i}{dt} V^j \Big) E_k, \qquad \text{if} \quad V = V^k E_k \tag{9}$$

Compare the formula we got by projecting to the tangent space If we project this to the tangent space we get

$$\frac{DV}{dt} = \big(\partial_j V^k + \Gamma_{ij}^j V^i \big) \frac{dx^j}{dt} f_k, \qquad \text{if} \quad V = V^k f_k, \tag{10}$$

where $f : \mathbf{R}^n \to M \subset \mathbf{R}^N$ is the embedding and

$$f_k = \partial_k f$$

One can of course express (10) in terms of the local coordinates $(x^1, \ldots, x^n)$ using the identification $V^k f_k = V^k E_k$. (and $dV^k/dt = \partial_j V^k \, dx^j/dt$.)

Geodesics.

A parameterized curve γ is called a **geodesic** if $\frac{D}{dt}\frac{d\gamma}{dt} = 0.$

If γ is a geodesic then $\frac{d}{dt}\Big\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt}\Big\rangle = 2\Big\langle \frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\Big\rangle = 0$. Therefore the arc length is proportional to the parameter for a geodesic.

In local coordinates $\gamma(t) = (x_1(t), \dots, x_n(t))$ the equation $\frac{D}{dt}\frac{d\gamma}{dt} = 0$ becomes

$$\frac{d^2x^k}{dt^2} + \Gamma_{ij}^k \frac{dx^i}{dt}\frac{dx^j}{dt} = 0, \qquad k = 1, \dots, n$$

We can think of this curve as a curve $(\gamma(t), d\gamma(t)/dt)$ in the tangent space $TM = \{(p, v);\, p \in M\, v \in T_pM\}$. In local coordinates $(x^1, \dots, x^n)$ a vector $v \in T_pM$ can be written $Y^i\partial_i$ so $(x^1, \dots, x^n, Y^1, \dots, Y^n)$ are local coordinates for TM. We get the system

$$\frac{dx^k}{dt} = Y^k, \qquad \frac{dY^k}{dt} - \Gamma_{ij}^k Y^i Y^j$$

By the existence theory for ODE this system has a unique local solution for given initial conditions for x and Y. However, because of the uniqueness a solution in a different coordinate system will in fact be the same which means that through each point in the tangent space TM there is a unique trajectory $(\gamma(t), \gamma'(t)) \in TM$. This vector field is called the **geodesic vector field** on TM.