

Lecture 11: Geodesics.

A parameterized curve γ is called a **geodesic** if

$$(1) \quad \frac{D}{dt} \frac{d\gamma}{dt} = 0.$$

In local coordinates $\gamma(t) = (x_1(t), \dots, x_n(t))$ this becomes

$$(2) \quad \frac{d^2 x^k}{dt^2} + \Gamma_{ij}^k \frac{dx^i}{dt} \frac{dx^j}{dt} = 0, \quad k = 1, \dots, n.$$

We consider this as the system

$$(3) \quad \frac{dx^k}{dt} = Y^k, \quad \frac{dY^k}{dt} = -\Gamma_{ij}^k Y^i Y^j.$$

This system of ODE has a unique local solution for given initial conditions

$$x|_{t=0} = x_0, \quad Y|_{t=0} = Y_0.$$

Moreover, given (x_1, Y_1) there is a $\varepsilon_1 > 0$ and a $\delta > 0$ such that the system has a unique solution for $|t| < \delta$ if $|x_0 - x_1| < \varepsilon_1$ and $|Y_0 - Y_1| < \varepsilon_1$.

Any curve $\gamma(t)$ in M determines a curve $(\gamma(t), \gamma'(t))$ in the tangent space TM . Local coordinates in TM ; $(x^1, \dots, x^n, Y^1, \dots, Y^n)$, are obtained by for a given local coordinate system (x^1, \dots, x^n) of M expressing the tangent vectors as $Y^k \partial / \partial x^k$. The integral curves of the **geodesic vector field**:

$$G = Y^k \frac{\partial}{\partial x^k} - \Gamma_{ij}^k Y^i Y^j \frac{\partial}{\partial Y^k}$$

on TM are exactly the solutions to (3), i.e geodesics on M . That G is invariantly defined on TM follows since if f is a function on TM and $\gamma(t)$ is a geodesic then

$$\frac{d}{dt} f(\gamma(t), \gamma'(t)) = Gf(\gamma(t), \gamma'(t)),$$

as is seen in local coordinates. The left hand side is clearly invariantly defined.

The existence theorem can also be stated in an invariant way:

Th Given $p \in M$ there is an open set $p \in V \subset M$, $\delta > 0$, $\varepsilon_1 > 0$ and a C^∞ mapping

$$\gamma : (-\delta, \delta) \times \mathcal{U} \rightarrow M, \quad \mathcal{U} = \{(q, v); q \in V, v \in T_q M, |v| < \varepsilon_1\}$$

such that the curve $t \rightarrow \gamma(t, q, v)$, $|t| < \delta$, is the unique geodesic which, at $t = 0$, passes through q with velocity v , for each $q \in V$ and for each $v \in T_q M$ with $|v| < \varepsilon_1$.

Rescaling lemma If $\gamma(t)$ is a geodesic then $\gamma(t, q, av) = \gamma(at, q, v)$.

Pf If $x(t)$ is the expression for $\gamma(t, q, v)$ in local coordinates then $y(t) = x(at)$ is a geodesic with $y(0) = x(0)$ and $\dot{y}(0) = a\dot{x}(0)$. In fact,

$$\frac{d^2 y^k(t)}{dt^2} + \Gamma_{ij}^k(y(t)) \frac{dy^i(t)}{dt} \frac{dy^j(t)}{dt} = a^2 \left(\frac{d^2 x^k}{dt^2}(at) + \Gamma_{ij}^k(x(at)) \frac{dx^i}{dt}(at) \frac{dx^j}{dt}(at) \right) = 0.$$

Using the rescaling lemma we can reformulate the existence theorem:

Th Given $p \in M$ there is an open set $p \in V \subset M$, $\delta > 0$, $\varepsilon > 0$ and a C^∞ mapping

$$\gamma : (-2, 2) \times \mathcal{U} \rightarrow M, \quad \mathcal{U} = \{(q, v); q \in V, v \in T_q M, |v| < \varepsilon\}$$

such that the curve $t \rightarrow \gamma(t, q, v)$, $|t| < 2$, is the unique geodesic which, at $t = 0$, passes through q with velocity v , for each $q \in V$ and for each $v \in T_q M$ with $|v| < \varepsilon$.

We now define the exponential map by

$$\exp(q, v) = \gamma(1, q, v), \quad (q, v) \in \mathcal{U}$$

and set $\exp_q(v) = \exp(q, v)$.

Th Given $q \in M$, there exists an $\varepsilon > 0$ such that $\exp_q(v) : B_\varepsilon(0) \subset TM \rightarrow M$, is a diffeomorphism of $B_\varepsilon(0)$ onto an open subset of M .

Pf

$$d(\exp_q)_o(v) = \left. \frac{d}{dt}(\exp_q tv) \right|_{t=0} = \left. \frac{d}{dt}(\gamma(1, q, tv)) \right|_{t=0} = \left. \frac{d}{dt}\gamma(t, q, v) \right|_{t=0} = v.$$

i.e. $d(\exp_q)_o$ is the identity map of $T_q M$, and it follows from the inverse function theorem that \exp_q is a local diffeomorphism.