Lecture 12: Mimimizing properties of geodesics.

A **piecewise differentiable curve** is a continuous curve $c:[a,b]\to M$ such that there is a partition $a=t_0<\ldots<t_k=b$ such that $c:[t_{i-1},t_i]\to M$ are differentiable. Parallel transport extends to piecewise differentiable curves.

A **parameterized surface** is a differentiable mapping $A\to M$, where $A=[a,b]\times[c,d]$. For each v_0, $u\to s(u,v_0)$ is a curve in M and $\partial s/\partial u \equiv ds(\partial/\partial u)$ is a vector field along this curve and hence along the parameterized surface. For a vector field V along the parameterized surface we define the covariant derivative $DV/\partial u$ as the covariant derivative along the curves $u\to s(u,v_0)$. Similarly for $\partial s/\partial v$ and $DV/\partial v$.

Symmetry lemma If M is a differentiable manifold with a symmetric connection and $s : A \to M$ is a parameterized surface then:

$$\frac{D}{\partial v}\frac{\partial s}{\partial u}=\frac{D}{\partial u}\frac{\partial s}{\partial v}$$

Rem Note that for an isometrically imbedded manifold the covariant derivative is just the orthogonal projection to the tangent plane of the derivative so the lemma then follows from the equality of mixed partial derivatives.
Pf If $\mathbf{x}:V\subset\mathbf{R}^n\to M$ are coordinates then $\mathbf{x}^{-1}\circ s(u,v)=(x^1(u,v),\ldots,x^n(u,v))$ and $\partial s/\partial u$ is the vector field which on functions is $\partial f\circ s(u,v)/\partial u=\partial f/\partial x^i\,\partial x^i/\partial u$ so

$$\frac{D}{\partial v}\Big(\frac{\partial s}{\partial u}\Big)=\frac{D}{\partial v}\Big(\frac{\partial x^i}{\partial u}\frac{\partial}{\partial x^i}\Big)=\frac{\partial^2 x^i}{\partial v\partial u}\frac{\partial}{\partial x^i}+\frac{\partial x^i}{\partial u}\nabla_{\frac{\partial x^j}{\partial v}\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}=\frac{\partial^2 x^i}{\partial v\partial u}\frac{\partial}{\partial x^i}+\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial v}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}$$

Since the connection is symmetric this is invariant under interchange of u and v.

In what follows we will identify the tangent space to T_pM at $v\in T_pM$ with T_pM.
Gauss lemma Let $p\in M$, $v\in T_pM$ such that $\exp_p v$ is defined. Let $w\in T_pM$. Then

$$\big\langle (d\exp_p)_v(v),(d\exp_p)_v(w)\big\rangle=\langle v,w\rangle.$$

Pf Note first that

$$(d\exp_p)_v(v)=\frac{d}{dt}\exp_p(tv)\Big|_{t=1}=\frac{d}{dt}\gamma(t,q,v)\Big|_{t=1}$$

and since $|\dot\gamma(t)|^2=|\dot\gamma(0)|^2=|v|^2$, the lemma follows if w is a multiple of v.
We can hence assume that $\langle v,w\rangle=0$. Since $\exp_p v$ is defined there is a $\varepsilon>0$ so that

$$f(t,s)=\exp_p(tv(s)),\qquad (t,s)\in A=\{(t,s);\,0\le t\le 1,\,-\varepsilon\le s\le\varepsilon\}$$

is defined if $v(s)$ is a curve in T_pM with $v(0)=v$, $\dot v(0)=w$, $|v(s)|=const$. We have

$$\text{(1)}\qquad \Big\langle\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\Big\rangle(1,0)=\big\langle (d\exp_p)_v(w),(d\exp_p)_v(v)\big\rangle$$

and

$$\frac{\partial}{\partial t}\Big\langle\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\Big\rangle=\Big\langle\frac{D}{\partial t}\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\Big\rangle+\Big\langle\frac{\partial f}{\partial s},\frac{D}{\partial t}\frac{\partial f}{\partial t}\Big\rangle$$

The last expression is zero since $\partial f/\partial t$ is the tangent vector of the geodesic. From the symmetry of the connection the first can be written

$$\Big\langle\frac{D}{\partial t}\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\Big\rangle=\Big\langle\frac{D}{\partial s}\frac{\partial f}{\partial t},\frac{\partial f}{\partial t}\Big\rangle=\frac{1}{2}\frac{\partial}{\partial s}\Big\langle\frac{\partial f}{\partial t},\frac{\partial f}{\partial t}\Big\rangle=0$$

Hence (1) is independent of t and the lemma follows since $\partial f/\partial s(0,0)=0$.