Lecture 13: Minimizing properties of geodesics.

If $\exp_p$ is a diffeomorphism of a neighborhood V of the origin in T_pM then $\exp_p V$ is called a **normal neighborhood** of p. We have shown that each point has a normal neighborhood.

Proposition Let U be a normal neighborhood of p and $B \subset U$ be a normal ball of center p. Let $\gamma : [0,1] \to B$ be a geodesic with $\gamma(0) = p$. If $c : [0,1] \to B$ is any piecewise differentiable curve joining $\gamma(0)$ to $\gamma(1)$ then $\ell(\gamma) \leq \ell(c)$ and if equality holds then $c = \gamma$. Here the arc length $\ell(\gamma) = \int_0^1 |\dot\gamma(t)|\, dt$.
Proof Suppose first that c is contained in B. We can write $c(t) = \exp_p u(t)$, and $u(t) = r(t)v(t)$, where $|v(t)| = 1$ and $r(t) = |u(t)|$. Hence $c(t) = f(r(t), t)$, where $f(r,t) = \exp_p(rv(t))$. We can assume that $c(t_1) \neq p$, for $t_1 > 0$, otherwise ignore the interval $[0, t_1)$. Hence

$$\frac{dc}{dt} = \frac{\partial f}{\partial r}\frac{dr}{dt} + \frac{\partial f}{\partial t}$$

From Gauss lemma $\langle \partial f/\partial r, \partial f/\partial t\rangle = 0$. Since $|\partial f/\partial r| = 1$,

$$\left|\frac{dc}{dt}\right|^2 = \left|\frac{dr}{dt}\right|^2 + \left|\frac{\partial f}{\partial t}\right|^2 \geq \left|\frac{dr}{dt}\right|^2$$

so

$$\ell(c) = \int_0^t \left|\frac{dc}{dt}\right| dt \geq \int_0^t \left|\frac{dr}{dt}\right| dt \geq \int_0^t \frac{dr}{dt}\, dt = r(1) - r(0) = \ell(\gamma).$$

If c is not contained in B, consider the curve restricted to $[0, t_1)$ where t_1 is the first time c leaves B.

Note that a geodesic is in general not globally minimizing, as is seen on the sphere. We shall prove that if a piecewise smooth curve is minimizing then it is a geodesic. For a smooth curve this follows from the existence of normal neighborhoods but for only piecewise smooth curves we need to show that every point has a neighborhood which is normal to all its points. This is called a **totally normal neighborhood**:
Theorem For any $p \in M$ there is a neighborhood W of p and a $\delta > 0$, such that, for every $q \in W$, $\exp_q$ is a diffeomorphism on $B_\delta(0) \subset T_qM$ and $\exp_q(B_\delta(0)) \supset W$.
Proof Let $\mathcal{U} = \{(q, v);\, q \in V,\, v \in T_qM,\, |v| < \varepsilon\}$ be such that $\exp_q v$ is defined and

$$F(q, v) = (q, \exp_q v)$$

Then $F(p, 0) = (p, p)$ and we claim that

$$dF_{(p,0)} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$$

In fact, since $(d\exp_p)_0 = I$ we have

$$dF_{(p,0)}(0, w) = \frac{d}{dt}(p, \exp_p(wt))\Big|_{t=0} = (0, w)$$
$$dF_{(p,0)}(\alpha'(0), 0) = \frac{d}{dt}(\alpha(t), \exp_{\alpha(t)}(0))\Big|_{t=0} = (\alpha'(0), \alpha'(0))$$

It follows that we can apply the inverse function theorem and conclude the existence of a neighborhood $\mathcal{U}' = \{(q, v);\ q \in V',\ v \in T_qM,\ |v| < \varepsilon'\}$, where $V' \subset V$ is a neighborhood of p, such that F maps $\mathcal{U}'$ diffeomorphically onto a neighborhood W' of (p, p). Let W be a neighborhood of p such that $W \times W \subset W'$.

Corollary If a piecewise differentiable curve $\gamma : [a, b] \to M$, with parameter proportional to arc length, has length less than or equal to the length of any other piecewise differentiable curve joining $\gamma(a)$ to $\gamma(b)$ then γ is a geodesic.

Convex neighborhoods.

A subset S is called **strongly convex** if for any two points q_1, q_2 in the closure $\overline{S}$, there is a unique minimizing geodesic γ joining q_1 to q_2 whose interior is in S.
Prop For any p there is a $\beta > 0$ such that the geodesic ball $B_\beta(p)$ is strongly convex.
Lemma for any p there is a $c > 0$ such that any geodesic that is tangent at q to the geodesic sphere $S_r(p)$ of radius $r < c$ stays out of the geodesic ball $B_r(p)$ for some neighborhood of q.