Lecture 14: Curvature.

For each pair of vector fields $X, Y \in \mathcal{X}$ the **curvature** is an operator on the vector fields $R(X,Y) : \mathcal{X} \to \mathcal{X}$:

$$R(X,Y) = \nabla_Y \nabla_X - \nabla_X \nabla_Y + \nabla_{[X,Y]}$$

It vanishes if the metric is Euclidean. Then in coordinates $\nabla_X Y = X(Y^\ell)\partial_\ell$, so $\nabla_Y \nabla_X Z = Y\big(X(Z^\ell)\big)\partial_\ell$ and hence $\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z = \big(YX - XY\big)(Z^\ell)\partial_\ell$.

Prop (f-linearity) If $f, g \in \mathcal{D}$ are functions then for any vector fields:

$$R(fX_1 + gX_2, Y) = fR(X_1, Y) + gR(X_2, Y),$$
$$R(X, fY_1 + gY_2) = fR(X, Y_1) + gR(X, Y_2),$$
$$R(X,Y)(fZ_1 + gZ_2) = fR(X,Y)Z_1 + gR(X,Y)Z_2$$

Prop (Bianchi Identity)

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

The proof uses that since the connection is symmetric $\nabla_X Y - \nabla_Y X = [X,Y]$.

With a slight abuse of notation let

$$R(X,Y,Z,T) = \langle R(X,Y)Z, T\rangle$$

Then we have the following symmetry properties:
Prop

$$
\begin{aligned}
&(a) \qquad R(X,Y,Z,T) + R(Y,Z,X,T) + R(Z,X,Y,T) = 0\\
&(b) \qquad R(X,Y,Z,T) = -R(Y,X,Z,T)\\
&(c) \qquad R(X,Y,Z,T) = -R(X,Y,T,Z)\\
&(d) \qquad R(X,Y,Z,T) = R(Z,T,X,Y)
\end{aligned}
$$

(a) is the Bianchi identity and (b) follows directly form the definition. (c) is equivalent to $R(X,Y,Z,Z) = 0$. (d) follows from (a).

It is convenient to express all these identities in coordinates. Let $X_i = \partial/\partial x^i$ and

$$R_{ijk\ell} = R(X_i, X_j, X_k, X_\ell), \qquad R(X_i, X_j)X_k = R_{ijk}^\ell X_\ell$$

Then the above proposition becomes

$$
\begin{aligned}
&R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0\\
&R_{ijk\ell} = -R_{jki\ell}\\
&R_{ijk\ell} = -R_{ij\ell k}\\
&R_{ijk\ell} = R_{k\ell ij}
\end{aligned}
$$

Recall that we defined $\nabla_{X_i} X_j = \Gamma_{ij}^\ell X_\ell$ and since $\nabla_X (fZ) = X(f)Z + f\nabla_X Z$;

$$
\begin{aligned}
R(X_i, X_j)X_k &= \nabla_{X_j}\nabla_{X_i} - \nabla_{X_i}\nabla_{X_j} = \nabla_{X_j}\big(\Gamma_{ik}^\ell X_\ell\big) - \nabla_{X_i}\big(\Gamma_{jk}^\ell X_\ell\big)\\
&= \Big(\frac{\partial}{\partial x^j}\Gamma_{ik}^\ell - \frac{\partial}{\partial x^i}\Gamma_{jk}^\ell + \Gamma_{ik}^m \Gamma_{jm}^\ell - \Gamma_{jk}^m \Gamma_{im}^\ell\Big) X_\ell = R_{ijk}^\ell X_\ell
\end{aligned}
$$