Lecture 15: Sectional curvature.

Prop Let $\sigma \subset T_pM$ be a two dimensional subspace and let $x, y \in \sigma$ be linearly independent. Then the **sectional curvature**

$$K(x,y) = \frac{R(x,y,x,y)}{|x \wedge y|^2}, \qquad \text{where} \quad |x \wedge y|^2 = |x|^2\,|y|^2 - \langle x, y\rangle^2$$

does not depend on the choice of $x, y \in \sigma$.
Pf It follows from $K(x,y) = K(y,x)$, $K(x,y) = K(\lambda x, y)$, $K(x,y) = K(x + \lambda y, y)$.

Lemma Suppose the $R(x,y,z,t)$ and $R'(x,y,z,t)$ satisfy

$$\begin{aligned}
&(a) \qquad R(X,Y,Z,T) + R(Y,Z,X,T) + R(Z,X,Y,T) = 0\\
&(b) \qquad R(X,Y,Z,T) = -R(Y,X,Z,T)\\
&(c) \qquad R(X,Y,Z,T) = -R(X,Y,T,Z)\\
&(d) \qquad R(X,Y,Z,T) = R(Z,T,X,Y)
\end{aligned}$$

Then if $R(x,y,x,y) = R'(x,y,x,y)$ it follows that $R(x,y,z,t) = R'(x,y,z,t)$.
Pf Since

$$R(x+z, y, x+z, y) = R'(x+z, y, x+z, y)$$

it follows that

$$R(x,y,x,y) + 2R(x,y,z,y) + R(z,y,z,y) = R'(x,y,x,y) + 2R'(x,y,z,y) + R'(z,y,z,y)$$

and hence

$$R(x,y,z,y) = R'(x,y,z,y)$$

It follows that

$$R(x, y+t, z, y+t) = R'(x, y+t, z, y+t)$$

and hence

$$R(x,y,z,t) + R(x,t,z,y) = R'(x,y,z,t) + R'(x,t,z,y)$$

which ca be written as

$$R(x,y,z,t) - R'(x,y,z,t) = R(y,z,x,t) - R'(y,z,x,t)$$

It follows that the expression is invariant under cyclic permutations and by (a)

$$3R(x,y,z,t) - 3R'(x,y,z,t) = 0$$

Corollary If M has constant sectional curvature then

$$R(X,Y,W,Z) = K\Big(\langle X, W\rangle\langle Y, Z\rangle - \langle Y, W\rangle\langle X, Z\rangle\Big).$$

Lemma Let $f : \mathbf{R}^2 \to M$ be a parameterized surface and let $V(s,t)$ be a vector field along $f(s,t)$. Let $D/\partial s$ be covariant differentiation along the curve $s \to f(s, t_0)$ and $D/\partial t$ be covariant differentiation along the curve $t \to f(s_0, t)$. Then

$$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = R\Big(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\Big)$$

We also gave the definitions of **Ricci curvature** and **scalar curvature**.