Lecture 16: Tensors. A **covariant tensor of order** r is a multilinear map

$$T : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to \mathcal{D}(M)$$

i.e. given r vector fields $Y_1, \ldots, Y_r$ on M, $T(Y_1, \ldots, Y_r)$ is a function on M, and T is f-linear in each argument

$$T(Y_1, \ldots fX + gY, \ldots, Y_r) = f\,T(Y_1, \ldots, X, \ldots, Y_r) + gT(Y_1, \ldots, Y, \ldots, Y_r)$$

A tensor is a pointwise object: Locally in a neighborhood of a point we can find a basis $E_1, \ldots, E_n \in \mathcal{X}(M)$, and expressing $Y_k = Y_k^{i_k} E_{i_k}$ we find that by f-linearity

$$T(Y_1, \ldots, Y_r) = Y_1^{i_1} \cdots Y_r^{i_r}\, T(E_{i_1}, \ldots, E_{i_r})$$

Here $T_{i_1 \ldots i_r} = T(E_{i_1}, \ldots, E_{i_r})$ are called the **components** of T in the frame $\{E_i\}$.

Ex The curvature considered as map $\mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{D}(M)$:

$$R(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle$$

is a tensor (we have showed that its f-linear in each argument).

Ex The connection ∇ considered as a map $\mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{D}(M)$:

$$\nabla(X, Y, Z) = \langle \nabla_X Y, Z\rangle$$

is not a tensor, since it is not f-linear in Y; $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

The **covariant differential** ∇T of a tensor T of order r is the tensor of order $r+1$:

$$\nabla T(Y_1, \ldots, Y_r, Z) = Z\big(T(Y_1, \ldots, Y_r)\big) - T(\nabla_Z Y_1, \ldots, Y_r) - \cdots - T(\nabla_Z Y_1, \ldots, \nabla_Z Y_r)$$

It follows directly from the definition that the covariant differential is f-linear. The **covariant derivative** $\nabla_Z T$ of T relative to $Z \in \mathcal{X}(M)$ is the tensor of order r:

$$\nabla_Z T(Y_1, \ldots, Y_r) = \nabla T(Y_1, \ldots, Y_r, Z)$$

Let $p \in M$ and $\alpha(t)$ be a curve with $\alpha(0) = p$, $\alpha'(t) = Z(\alpha(t))$. Let $\{e_1, \ldots, e_n\}$ be a basis for T_pM and let $e_i(t)$ be the parallel transport of e_i along $\alpha(t)$. Let $T_{i_1 \ldots i_r} = T(e_{i_1}, \ldots, e_{i_r})$ be the components in the basis $\{e_i(t)\}$ along $\alpha(t)$. Then

$$\begin{aligned}&(\nabla_Z T)\big(e_{i_1}(t), \ldots, e_{i_r}(t)\big)\\ &\qquad = \frac{d}{dt} T_{i_1 \ldots i_r} - T(\nabla_Z e_{i_1}(t), \ldots, e_{i_r}(t)) - \cdots - T(e_{i_1}(t), \ldots, \nabla_Z e_{i_r}(t))\end{aligned}$$

Here $\nabla_Z e_i(t) = 0$ so

$$(\nabla_Z T)\big(e_{i_1}(t), \ldots, e_{i_r}(t)\big) = \frac{d}{dt} T_{i_1 \ldots i_r}$$

Ex We can identify a vector field $X \in \mathcal{X}(M)$ with a tensor $\mathcal{X}(M) \to \mathcal{D}(M)$ by

$$X(Y) = \langle X, Y\rangle$$

The covariant derivative of the tensor $X : \mathcal{X}(M) \to \mathcal{D}(M)$ with respect to Z is

$$\nabla_Z X(Y) = Z(X(Y)) - X(\nabla_Z Y) = Z\langle X, Y\rangle - \langle X, \nabla_Z Y\rangle = \langle \nabla_Z X, Y\rangle,$$

i.e. the tensor $\nabla_Z X$ can be identified with the vector $\nabla_Z X$.

Ex The covariant derivative of the metric tensor $G : X, Y \to \langle X, Y\rangle$, is 0:

$$\nabla G(X, Y, Z) = Z\langle X, Y\rangle - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle = 0.$$