Lecture 18: Isometric immersions.

Let $f : M \to \overline{M}$ be a differentiable immersion of a manifold M of dimension n into a Riemannian manifold $\overline{M}$ of dimension $n+m$ (i.e. df_p is injective.)
The Riemannian metric on $\overline{M}$ induces a metric on M: for $v_1, v_2 \in T_pM$ define $\langle v_1, v_2\rangle = \langle df_p(v_1), df_p(v_2)\rangle$. Then f becomes an isometric immersion of M into $\overline{M}$.

In the case of a surface S in $\mathbf{R}^3$ which is a graph $z = f(x,y)$ of a function with $f(0,0) = f_x(0,0) = f_y(0,0) = 0$, S close to $\mathbf{0} \in \mathbf{R}^3$ looks like $z = II(x,y)/2$, where

$$II(x,y) = f_{xx}\, x^2 + 2f_{xy}(0,0)\, xy + f_{yy}(0,0)\, y^2$$

II is called the second fundamental form of S. The Gaussian curvature is given by

$$K = f_{xx}f_{yy} - f_{xy}^2.$$

We want to give the general definition of the second fundamental form of M in $\overline{M}$. We will see that there is a relation between the curvature on M and $\overline{M}$ and the second fundamental form generalizing the above formula.

The second fundamental form. We have

$$T_p\overline{M} = T_pM \oplus (T_pM)^\perp$$

where $(T_pM)^\perp$ is the orthogonal complement of T_pM in $T_p\overline{M}$, i.e. if $v \in T_p\overline{M}$ we can write in a unique way

$$v = v^T + v^N, \qquad v^T \in T_pM, \quad v^N \in (T_pM)^\perp$$

We call v^T the **tangential component** of v and v^N the **normal component**.
The Riemannian connection on $\overline{M}$ will be denoted by $\overline{\nabla}$ and that of M by ∇.
If X, Y are vector fields on M and $\overline{X}, \overline{Y}$ are local extensions to $\overline{M}$ we claim that

$$\nabla_X Y = (\overline{\nabla}_{\overline{X}}\overline{Y})^T$$

In fact, its a symmetric connection compatible with the metric and as such unique.
If X, Y are vector fields on M then

$$B(X,Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y$$

is vector field on $\overline{N}$ normal to M. $B(X,Y)$ is independent of the extensions $\overline{X}, \overline{Y}$ and hence is well defined. Let $\mathcal{X}(M)^\perp$ denote the vector fields normal to M.

Prop The mapping $B : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)^\perp$ is bilinear and symmetric.
Pf The symmetry follows from the symmetric of the Riemannian connection

$$\overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y = \overline{\nabla}_{\overline{Y}}\overline{X} + \left[\overline{X}, \overline{Y}\right] - \nabla_Y X - [X,Y] = \overline{\nabla}_{\overline{Y}}\overline{X} - \nabla_Y X$$

The **second fundamental form** along the normal vector $\eta \in (T_pM)^\perp$ is the quadratic form:

$$II_\eta(x) = H_\eta(x,x), \qquad \text{where} \quad H_\eta(x,y) = \langle B(x,y), \eta\rangle.$$

Associated with it is a self-adjoint operator $S_\eta : T_pM \to T_pM$ given by

$$\langle S_\eta(x), y\rangle = H_\eta(x,y)$$

Prop Let N be a local extension of the normal η. Then $S_\eta(x) = -(\overline{\nabla}_x N)^T$.
Pf Since $\langle \overline{Y}, N\rangle = 0$ and $\langle \nabla_X Y, N\rangle = 0$ it follows that

$$\langle S_\eta(X), Y\rangle = \langle B(\overline{X}, \overline{Y}), N\rangle = \langle \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y, N\rangle = \langle \overline{\nabla}_{\overline{X}}\overline{Y}, N\rangle = -\langle \overline{Y}, \overline{\nabla}_{\overline{X}} N\rangle.$$