

## Lecture 19.

Let  $M$  be a submanifold of a Riemannian manifold  $\overline{M}$  with induced metric on  $M$ . Let  $\nabla$  be the connection on  $M$  and  $\overline{\nabla}$  be the connection on  $\overline{M}$ . Then  $\nabla_X Y = (\overline{\nabla}_{\overline{X}} \overline{Y})^T$ , where  $\overline{X}, \overline{Y}$  are extensions to  $\overline{M}$  of the vector fields  $X, Y$  on  $M$ , and the tangential component  $W^T$  stands for the orthogonal projection of  $T_p \overline{M}$  to  $T_p M$ . The second fundamental form is given by  $B(X, Y) = \overline{\nabla}_{\overline{X}} \overline{Y} - \nabla_X Y$ , or if  $\eta$  is a normal vector field  $H_\eta(X, Y) = \langle B(X, Y), \eta \rangle = \langle S_\eta(X), Y \rangle$ , where  $S_\eta$  is selfadjoint operator given by  $S_\eta(X) = -(\overline{\nabla}_X N)^T$ , where  $N$  is an extension of  $\eta$  to  $\overline{M}$ .

**Th (Gauss)** Let  $K$  be the sectional curvature on  $M$  and  $\overline{K}$  the sectional curvature on  $\overline{M}$ . Then

$$K(X, Y) - \overline{K}(X, Y) = \langle B(X, X), B(Y, Y) \rangle - |B(X, Y)|^2$$

Before we give the proof let us look at a couple of special cases. If  $\overline{M} = \mathbf{R}^N$  then  $\overline{K} = 0$  and this in particular implies the famous Theorem Egregium of Gauss, which says that the Gaussian curvature of a surface in space is an invariant under isometries. On the other hand the theorem above also give an interpretation of the sectional curvature of  $\overline{M}$ . Namely, let  $\sigma \subset T_p \overline{M}$  be a two plane and let  $M$  be the image of the exponential map of a neighborhood of the origin on  $\sigma$ . Then we claim that the second fundamental form of  $M$  vanishes at the  $p$ . In fact if  $\eta$  is a normal vector and if  $N$  is an extension of  $\eta$  and  $X$  is an extension of a vector in  $\sigma$  such that  $\langle N, X \rangle = 0$  then

$$H_\eta(X) = \langle S_\eta(X), X \rangle - \langle \overline{\nabla}_X N, X \rangle = -X \langle N, X \rangle + \langle N, \overline{\nabla}_X X \rangle = 0$$

since  $X$  is an extension of the velocity vector field of a geodesic.

We proceeded with proving the theorem above as in the book.