

## Lecture 2: Curvature of submanifolds of Euclidean space.

Let  $M$  be a  $C^\mu$  **submanifold** of dimension  $n$  in  $\mathbf{R}^N$ . This means that for every  $p \in M$  there is a neighborhood  $U$  of  $p$  in  $\mathbf{R}^N$  and a neighborhood  $\Omega$  of  $0$  in  $\mathbf{R}^n$  and a map  $f \in C^\mu(\Omega, \mathbf{R}^N)$ , which is a bijection  $f: \Omega \rightarrow M \cap U$  with injective differential. The map is called a **local parametrization**. Note that there are many local parameterizations, in fact if  $\phi: \Omega' \rightarrow \Omega$  is a diffeomorphism then  $f \circ \phi$  is another.

If  $I \ni t \rightarrow X(t) \in M$  is a  $C^\mu$  curve in  $M$  and  $f$  a parametrization near  $X_0 = X(t_0)$  then since  $f$  is invertible we can write  $X(t) = f(x(t))$  where  $x(t) \in \Omega$ , for  $t$  close to  $t_0$ . The **tangent vector** to the curve is

$$\frac{dX}{dt} = \frac{df(x)}{dt} = f'(x) \frac{dx}{dt} = \frac{dx^j}{dt} f_j(x), \quad \text{where} \quad f_j(x) = \frac{\partial f(x)}{\partial x^j} = \partial_j f(x)$$

and we used the convention that repeated upper and lower indices are summed over. The tangent vectors of all  $C^1$  curves passing through  $f(x)$  span the **tangent plane**:

$$T_{f(x)}M = f'(x)\mathbf{R}^n$$

The tangent plane is independent of the particular choice of  $f$ .

The arc length  $s$  on along the curve  $x(t)$  is given by

$$\left(\frac{ds}{dt}\right)^2 = \left\langle \frac{dX}{dt}, \frac{dX}{dt} \right\rangle = g_{ij}(x) \frac{dx^i}{dt} \frac{dx^j}{dt}, \quad \text{where} \quad g_{ij}(x) = \langle f_j(x), f_k(x) \rangle$$

This can be considered as quadratic form on the tangent space  $T_{f(x)}$ , called the **first fundamental form**, since it is the restriction of the Euclidean metric in  $\mathbf{R}^N$  to the tangent space  $T_{f(x)}M$ , and so is independent of the particular parametrization  $f$ .

Let us now look at the second derivative of the curve:

$$\frac{d^2 f(x(t))}{dt^2} = \frac{d^2 x^j}{dt^2} f_j(x(t)) + \frac{dx^i}{dt} \frac{dx^j}{dt} f_{jk}(x(t)), \quad \text{where} \quad f_{ij}(x) = \partial_i \partial_j f(x)$$

The first sum is in the tangent plane and if we project on the normal plane  $N_{f(x)}M$  to  $T_{f(x)}M$  (i.e. all vectors orthogonal to  $T_{f(x)}$ ) we can eliminate this sum and obtain: Let  $h_{ij}(x)$  be the orthogonal projection of  $f_{ij}(x)$  in the normal plane  $N_{f(x)}$ . The sum

$$h_{ij}(x) \frac{dx^i}{dt} \frac{dx^j}{dt}$$

can be thought of as quadratic form on  $T_{f(x)}M$  with values in  $N_{f(x)}$  independent of the particular choice of parametrization  $f$  since it is the projection of  $\ddot{X}$  and since  $\mathbf{R}^n \ni \dot{x} \rightarrow f'(x)\dot{x} = \dot{X} \in T_{f(x)}M$  is a bijection. It is called the **second fundamental form**. Classically it was derived for hypersurfaces  $n = N - 1$  in which case it is a function times the normal. It measures how the hypersurface is curving in the surrounding space. It is the second order derivatives of the hypersurface measured in a way which is independent of the parametrization, similar to how the curvature of a curve was defined so it did not depend on a particular parametrization.

**Th** For every  $C^2$  curve on  $M$  with the unit tangent vector  $f'(x)\dot{x}$  at  $f(x)$  the sum above is the curvature of the curve times the orthogonal projection of the principal normal of the curve (defined in the previous section) in  $N_{f(x)}$ .