Lecture 20: The fundamental surface equations.

Let M be a submanifold of a Riemannian manifold $\overline{M}$ with induced metric on M. Let ∇ be the connection on M and $\overline{\nabla}$ be the connection on $\overline{M}$. Then $\nabla_X Y = (\overline{\nabla}_{\overline{X}}\overline{Y})^T$, where $\overline{X}, \overline{Y}$ are extensions to $\overline{M}$ of the vector fields X, Y on M, and the tangential component W^T stands for the orthogonal projection of $T_p\overline{M}$ to T_pM. The second fundamental form is given by $B(X,Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y$, or if η is a normal vector field $H_\eta(X,Y) = \langle B(X,Y), \eta\rangle = \langle S_\eta(X), Y\rangle$, where S_η is selfadjoint operator given by $S_\eta(X) = -(\overline{\nabla}_X N)^T$, where N is an extension of η to $\overline{M}$.

We now plan to study the normal component of $\overline{\nabla}_X \eta$, which will be called the **normal connection**:

$$\nabla^\perp_X \eta = (\overline{\nabla}_X \eta)^N = \overline{\nabla}_X \eta - (\overline{\nabla}_X \eta)^T = \overline{\nabla}_X \eta + S_\eta(X)$$

We define the normal curvature

$$R^\perp(X,Y)\eta = \nabla^\perp_Y \nabla^\perp_X \eta - \nabla^\perp_X \nabla^\perp_Y \eta + \nabla^\perp_{[X,Y]}\eta$$

Gauss equation:

$$\langle \overline{R}(X,Y)Z, T\rangle = \langle R(X,Y)Z, T\rangle - \langle B(Y,T), B(X,Y)\rangle + B(X,T), B(Y,Z)\rangle$$

Ricci equation

$$\langle \overline{R}(X,Y)\eta, \xi\rangle - \langle R^\perp(X,Y)\eta, \xi\rangle = \langle [S_\eta, S_\xi]X, Y\rangle, \qquad [S_\eta, S_\xi] = S_\eta \circ S_\xi - S_\xi \circ S_\eta$$

Let

$$B(X,Y,\eta) = \langle B(X,Y), \eta\rangle$$

and

$$(\overline{\nabla}_X B)(X,Y,\eta) = X\big(B(Y,Z,\eta)\big) - B(\nabla_X Y, Z, \eta) - B(Y, \nabla_X Z, \eta) - B(Y, Z, \nabla^\perp_X \eta)$$

Codazzi equation:

$$\langle \overline{R}(X,Y)Z, \eta\rangle = (\overline{\nabla}_Y B)(X,Z,\eta) - (\overline{\nabla}_X B)(Y,Z,\eta)$$

We did the proofs of these equations as in the book.