**Lecture 21: Cartan formalism.**

Let $E_1, \ldots, E_n$ be an orthonormal frame for $T_pM$, depending on $p \in U \subset M$ and let $\theta^1, \ldots, \theta^n$ be the dual frame of one forms, i.e. $\theta^i(E_j) = \delta_{ij}$. We have

**Prop** There are one forms $\omega^i_{\ j}$, such that

$$d\theta^i = -\omega^i_{\ j} \wedge \theta^j, \qquad \omega^i_{\ j} = -\omega^j_{\ i}.$$

**Pf** We define $\omega^i_{\ j}$ by

$$\nabla_X E_j = \omega^i_{\ j}(X) E_i$$

Since $\nabla_X g = 0$ if $g$ is the metric tensor it follow that

$$0 = g(\nabla_X E_i, E_j) + g(E_i, \nabla_X E_j) = \omega^j_{\ i}(X) + \omega^i_{\ j}(X).$$

By the invariant definition of the exterior derivative in terms of the directional derivative $D_X f = X(f) = X^\ell \partial_\ell f$,

$$d\theta^i(E_k, E_\ell) = D_{E_k}\theta^i(E_\ell) - D_{E_\ell}\theta^i(E_k) - \theta^i([E_k, E_\ell]) = -\theta^i([E_k, E_\ell]).$$

On the other hand

$$\begin{aligned} -(\omega^i_{\ j} \wedge \theta^j)(E_k, E_\ell) &= -\omega^i_{\ j}(E_k)\theta^j(E_\ell) + \omega^i_{\ j}(E_\ell)\theta^j(E_k) = -\omega^i_{\ \ell}(E_k) + \omega^i_{\ k}(E_\ell) \\ &= \theta^i(\nabla_{E_\ell} E_k) - \theta^i(\nabla_{E_k} E_\ell) = -\theta^i([E_k, E_\ell]) \end{aligned}$$

**Prop** The equations

$$\Omega^i_{\ j} = d\omega^i_{\ j} + \omega^i_{\ k} \wedge \omega^k_{\ j}$$

defines a skew-symmetric matrix of 2-forms that also gives the curvature tensor via

$$R(X,Y)E_j = \Omega^i_{\ j}(X,Y) \cdot E_i$$

**Pf** Recall that the curvature is defined by

$$R(X,Y)E_j = \nabla_X \nabla_Y E_j - \nabla_Y \nabla_X E_j - \nabla_{[X,Y]} E_j.$$

We have

$$\begin{aligned} & d\omega^i_{\ j}(X,Y) \cdot E_i + \omega^i_k \wedge \omega^k_j(X,Y) \cdot E_i \\ &= \big(\nabla_X \omega^i_{\ j}(Y) \cdot E_i - \nabla_Y \omega^i_{\ j}(X) \cdot E_i - \omega^i_{\ j}([X,Y]) \cdot E_i + \omega^i_k(X) \cdot E_i \cdot \omega^k_j(Y) - \omega^i_k(Y) \cdot E_i \cdot \omega^k_j(X) \\ &= \nabla_X \nabla_Y E_j - \omega^i_{\ j}(Y) \nabla_X E_i - \nabla_Y \nabla_X E_j + \omega^i_{\ j}(X) \nabla_Y E_i - \nabla_{[X,Y]} E_j \\ &\quad + \omega^k_{\ j}(Y) \nabla_X E_k - \omega^k_{\ j}(X) \nabla_Y E_k \\ &= R(X,Y)E_i \end{aligned}$$