Lecture 3:.

Recall that $f : \mathbf{R}^n \to M \subset \mathbf{R}^N$ is a local parametrization of a submanifold M, $f_i = \partial_i f$, $f_{ij} = \partial_i \partial_j f$. Since $f_1, ..., f_n$ form a basis for $T_{f(x)}$ we can write

$$f_{ik}(x) = \Gamma^{\ell}_{ik}(x)\, f_{\ell}(x) + h_{ik}(x),$$

where $h_{ik}(x)$ is the orthogonal projection of f_{ik} in the normal plane $N_{f(x)}$.
To calculate the coefficients Γ^{ℓ}_{ik} we take the scalar product with f_{ℓ}:

$$\Gamma^{\ell}_{ik} g_{\ell j} = \langle f_{ik}, f_j \rangle$$

By a miracle one can compute the right hand side by means of the coefficients of the first fundamental form $g_{ij} = \langle f_i(x), f_j(x) \rangle$ and its derivatives, for we have

$$\partial_k g_{ij} = \langle f_{ik}, f_j \rangle + \langle f_i, f_{jk} \rangle, \quad \partial_i g_{jk} = \langle f_{ji}, f_k \rangle + \langle f_j, f_{ki} \rangle, \quad \partial_j g_{ki} = \langle f_{kj}, f_i \rangle + \langle f_k, f_{ij} \rangle$$

If we add the first two equations and subtract the third we get

$$\Gamma_{ikj} = \langle f_{ik}, f_j \rangle = \frac{1}{2}\big(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ki}\big)$$

Note the symmetry in the indices i,k, and that $\partial_k g_{ij} = \Gamma_{ikj} + \Gamma_{jki}$.
If (g^{ij}) denotes the inverse of (g_{ij}) we obtain

$$\Gamma^{\ell}_{ik} = g^{\ell j} \Gamma_{ikj}$$

The functions Γ^{ℓ}_{ik} are called the **Christoffel symbols**.

Let $X(t) = f(x(t))$ be a curve on M. Then, since

$$\frac{d^2 f(x)}{dt^2} = \frac{d}{dt}\Big(\frac{dx^j}{dt} f_j(x)\Big) = \frac{d^2 x^j}{dt^2} f_j(x) + \frac{dx^i}{dt}\frac{dx^j}{dt} f_{jk}(x),$$

we have shown that

$$\frac{d^2 f(x)}{dt^2} = \Big(\frac{d^2 x^j}{dt^2} + \Gamma^j_{ik}\frac{dx^i}{dt}\frac{dx^k}{dt}\Big) f_j(x) + \frac{dx^i}{dt}\frac{dx^k}{dt} h_{ik}(x).$$

Th Suppose that $x(t)$ is parameterized by arc length, $\langle \dot{X}, \dot{X} \rangle = g_{ij}\dot{x}^i \dot{x}^j = 1$. Then the curvature times the orthogonal projection of the principal normal in $T_{f(x)}M$ is

$$\Big(\frac{d^2 x^j}{dt^2} + \Gamma^j_{ik}\frac{dx^j}{dt}\frac{dx^k}{dt}\Big) f_j(x)$$

Thus this vector does not depend on the choice of parametrization f. Its length is called the geodesic curvature and the direction is called the geodesic principal normal direction.

Covariant derivative.

Let V be a vector field on M, i.e. to each $X \in M \subset \mathbf{R}^N$ we have a vector $V(X) \in T_X M$ tangential to M. The **covariant derivative of V along the curve $X(t)$** is

$$\frac{DV(X(t))}{dt} \equiv \text{Orthogonal projection of } \frac{dV(X(t))}{dt} \text{ onto the tangent space } T_{X(t)}M$$

This is defined without any parametrization. In terms of a parametrization $f : \mathbf{R}^n \to M$ the tangent space at $f(x)$ is spanned by $f_j(x) = \partial_j f(x)$ and we can write the vector field as

$$V(f(x)) = v(x) = v^j(x) f_j(x)$$

If we also write the curve in terms of the parametrization $X(t) = f(x(t))$ we get

$$\frac{dv(x)}{dt} = \big(\partial_k v^j(x)\ f_j(x) + v^j(x) f_{jk}(x)\big)\frac{dx^k}{dt}$$

If we project this to the tangent space we get

$$\frac{DV}{dt} = \big(\partial_k v^j + \Gamma^j_{ik} v^i\big)\frac{dx^k}{dt} f_j$$

This is of course independent of the particular parametrization since it is defined without the use of a parametrization. The remarkable fact is that this only depends on the embedding $f : \mathbf{R}^n \to M \subset \mathbf{R}^N$ through the first fundamental form. This means that if $\tilde{f} : \mathbf{R}^n \to \tilde{M} \subset \mathbf{R}^{\tilde{N}}$ is another embedding with the same first fundamental form at $f(x)$ respectively $\tilde{f}(x)$ then the expressions for the covariant derivatives above in terms of the coefficients in front of the basis vectors are the same.