Lecture 4. Let us now determine the tangential components of derivatives of a normal vector field $n(x)$:

$$\text{Projection of } \partial_i n(x) \text{ to the tangent space} = n_i{}^k f_k$$

Since differentiation of the equation $\langle n, f_j \rangle = 0$ gives

$$\langle \partial_i n, f_j \rangle + \langle n, f_{ij} \rangle, \qquad \text{that is} \quad n_i{}^k g_{kj} = -\langle h_{ij}, n \rangle$$

Multiplying by the inverse g^{kj} gives

$$n_i{}^k = -\langle h_{ij}, n \rangle \, g^{kj}$$

Actually, if in addition its a hypersurface and n is the unit normal $\langle n, n \rangle = 1$ then $\langle \partial_i n, n \rangle = 0$ so that its already tangential.

It is now easy to calculate the tangential components of $f_{ijk} = \partial_i \partial_j \partial_k f$. Since $f_{ij} = \Gamma_{ij}^{\ell} f_\ell + h_{ij}$ we obtain

$$f_{kij} = \Big(\partial_k \Gamma_{ij}^m + \Gamma_{ij}^{\ell} \Gamma_{\ell k}^m - \langle h_{ij}, h_{k\ell} \rangle g^{\ell m} \Big) f_m \quad \text{mod } N_{f(x)}.$$

Since $f_{kij} - f_{jik} = 0$ a calculation shows that:
Th The Gauss equations The first and second fundamental form are related by

$$R^m{}_{ijk} \equiv \partial_j \Gamma_{ik}^m - \partial_k \Gamma_{ij}^m + \Gamma_{ik}^{\ell} \Gamma_{\ell j}^m - \Gamma_{ij}^{\ell} \Gamma_{\ell k}^m = \big(\langle h_{ik}, h_{j\ell} \rangle - \langle h_{ij}, h_{k\ell} \rangle \big) g^{\ell m}.$$

Here the first equality is a definition. If we introduce $R_{\ell ijk} = g_{\ell m} R^m{}_{ijk}$ this can also be written

$$R_{\ell ijk} = \langle h_{ik}, h_{j\ell} \rangle - \langle h_{ij}, h_{k\ell} \rangle$$

Since the second fundamental form is defined invariantly (independently of a particular parametrization) on the tangent space so so is the **Riemann curvature tensor** defined by

$$R(T_1, T_2, T_3, T_4) = R_{ijk\ell} T_1^i T_2^j T_3^k T_4^\ell = \langle H(T_1, T_3), H(T_2, T_4) \rangle - \langle H(T_1, T_4), H(T_2, T_3) \rangle$$

where $H(T_1, T_2) = h_{ij} T_1^i T_2^j$.

We define the Ricci curvature and the scalar curvature by

$$R_{j\ell} = g^{ik} R_{ijk\ell}, \qquad R = g^{j\ell} R_{j\ell}$$

respectively. These are also invariantly defined (why?).