Lecture 5.

Suppose now that M is a hypersurface in $\mathbf{R}^N$, i.e. $N = n+1$ where n is the dimension of the submanifold M. Then we can write

$$h_{ij} = \tilde{h}_{ij}\, n$$

where n is a unit normal and $\tilde{h}_{ij}$ is scalar valued. Let $g(T,S) = g_{ij}T^iS^j$, $\tilde{h}(T,S) = \tilde{h}_{ij}T^iS^j$. The quotient

$$\frac{\tilde{h}(T,T)}{g(T,T)}$$

is the curvature of a curve on M with tangent vector $f'(x)T$ and principal normal in the direction of the normal to the surface, i.e. with vanishing geodesic curvature.

We can diagonalize the first and second fundamental forms simultaneously;

$$\tilde{h}(e_k, e_m) = K_k\delta_{km}, \qquad g(e_k, e_m) = \delta_{km}$$

In fact, since G is symmetric matrix we can diagonalize it and since it is positive definite we can take the square root and obtain: $G = A^TA$. Next we find the eigenvalues of $\tilde{H}$ with respect to G: $\tilde{H}e_k = K_kGe_k = K_kA^TAe_k$ which then is equivalent to $(A^{-1})^T\tilde{H}A^{-1}Ae_k = K_kAe_k$, which is symmetric so we can diagonalize it with an orthonormal basis $f_k = Ae_k$. Then $g(e_k.e_m) = \langle A^TAe_k, e_m\rangle = \langle Ae_k, Ae_m\rangle = \delta_{km}$.

The quotient of the second fundamental form divided by the first for a curve with vanishing geodesic curvature hence is maximized by the largest eigenvalue K_n and minimized by the smallest. We therefore call K_i the principal curvatures and e_i the principal curvature directions.

It follows from (*) and the discussion there that if n is the unit normal to a hypersurface then

$$\partial_i n = -\tilde{h}_{ij}g^{jk}f_k,$$

Thus

$$\langle T^i\partial_i n, S^kf_k\rangle = -\tilde{h}_{ij}T^iS^j$$

If we define the Gauss map by $M \ni f(x) \to n(x) \in \mathbf{S}^n$, then the differential $d\gamma : T_pM \to T_{n(p)}\mathbf{S}^n = T_pM$ and the above can be formulated

$$\langle dn(T), S\rangle = -\tilde{h}(T,S)$$

(Recall that the differential of a map is defined as follows. If $\alpha(t)$ is a curve with $\alpha'(0) = w$, let $\beta(t) = \gamma \circ \alpha(t)$. Then $d\gamma(w) = \beta'(0) = w^i\partial_i\gamma$.)
Hence the differential of the Gauss map is the linear transformation that corresponds to second fundamental form. In some modern books this is called the shape operator.