Lecture 6.

If we denote $(S,T)=g(S,T)$ and write $S=\sum_{j=1}^n (S,e_j)e_j$, $T=\sum_{i=1}^n (T,e_i)e_i$ we get

$$\tilde h(S,T) = \sum_{j=1}^k K_j (S,e_j)(T,e_j)$$

Since $h = \tilde h\, n$ it follows from Gauss equations that

$$\begin{aligned} R(T_1,T_2,T_3,T_4) &= \tilde h(T_1,T_3)\,\tilde h(T_2,T_4) - \tilde h(T_1,T_4)\,\tilde h(T_2,T_3) \\ &= \sum_{i\neq j} K_i K_j \Big((T_1,e_i)(T_3,e_i)(T_2,e_j)(T_4,e_j) - (T_1,e_i)(T_4,e_i)(T_2,e_j)(T_3,e_j) \Big) \\ &= \frac{1}{2}\sum_{i\neq j} K_i K_j \Big((T_1,e_i)(T_3,e_i)(T_2,e_j)(T_4,e_j) - (T_1,e_i)(T_4,e_i)(T_2,e_j)(T_3,e_j) \Big) \\ &+ \frac{1}{2}\sum_{i\neq j} K_i K_j \Big((T_1,e_j)(T_3,e_j)(T_2,e_i)(T_4,e_i) - (T_1,e_j)(T_4,e_j)(T_2,e_i)(T_3,e_i) \Big) \end{aligned}$$

Hence

$$R(T_1,T_2,T_3,T_4) = \frac{1}{2}\sum_{i\neq j} K_i K_j \begin{vmatrix} (T_1,e_i) & (T_1,e_j) \\ (T_2,e_i) & (T_2,e_j) \end{vmatrix} \begin{vmatrix} (T_3,e_i) & (T_3,e_j) \\ (T_4,e_i) & (T_4,e_j) \end{vmatrix}$$

If $n=2$ we conclude that $R(T_1,T_2,T_3,T_4)$ is K_1K_2 times the area of the parallelogram spanned by T_1,T_2 times the area of the parallelogram spanned by T_3,T_4. Hence if T,S are any two orthonormal unit vectors then $R(S,T,S,T) = K_1K_2$. Since the curvature is determined from the first fundamental form we have proven:

Teorema egregium of Gauss The total curvature K_1K_2 is determined by the first fundamental form.

Recall that the curvature tensor

$$R_{ijk\ell} = \langle h_{ik}, h_{j\ell}\rangle - \langle h_{i\ell}, h_{jk}\rangle, \qquad h_{ij} = h_{ji},$$

is antisymmetric in the pairs (i,j), (k,ℓ) but symmetric under exchange of the pairs

$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij}.$$

By expanding in a basis one sees that

$$R(S,T,S,T)(g(S,S)\,g(T,T) - g(S,T)^2)^{-1}$$

only depends on the plane spanned by S,T, and is therefore called the sectional curvature of the two plane. It turns out that the curvature tensor is determined by the sectional curvatures.