Lecture 7: Abstract manifolds and tangent space.

A **differentiable manifold** of dimension n is a is a Hausdorff topological space M with a countable basis of open sets and a family of injective mappings $\mathbf{x}_\alpha : U_\alpha \subset \mathbf{R}^n \to M$ of open sets U_α such that

(1) $\cup_\alpha \mathbf{x}_\alpha(U_\alpha) = M$

(2) for any α, β, with $\mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta) = W$ nonempty, the sets $\mathbf{x}_\alpha^{-1}(W)$ and $\mathbf{x}_\beta^{-1}(W)$ are open and $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$ is differentiable.

(3) The family $\{(U_\alpha, \mathbf{x}_\alpha)\}$ is maximal relative to the conditions (1) and (2).

The pair $(U_\alpha, \mathbf{x}_\beta)$ with $\mathbf{x}_\alpha(U_\alpha) \ni p$ is called a **(local) parametrization** at p.

The difference between this definition of an abstract manifold and an submanifold is that we are not assuming that M is a subset of some $\mathbf{R}^N$ and because of this there a priori is no meaning of differentiable functions on M and we only assume that the transition functions are differentiable. However, its a theorem that any manifold of dimension n can be embedded in $\mathbf{R}^{2n+1}$ so we could assume that to be the case and the definition would not use the structure of $\mathbf{R}^{2n+1}$.

A mapping $\varphi : M \to N$ between two differentiable manifolds is called **differentiable** at $p \in M$ if given a parametrization $\mathbf{y} : V \subset \mathbf{R}^m \to N$ at $\varphi(p)$ there is a parametrization $\mathbf{x} : U \subset \mathbf{R}^n \to M$ at p such that $\varphi(\mathbf{x}(U)) \subset \mathbf{y}(V)$ and the mapping

$$\mathbf{y}^{-1} \circ \varphi \circ \mathbf{x} : U \subset \mathbf{R}^n \to \mathbf{R}^m$$

is differentiable.

Let M be a differentiable manifold. A differentiable function $\alpha : (-\varepsilon, \varepsilon) \to M$ is a called a **differentiable curve** in M. Suppose that $\alpha(0) = p \in M$, and let $\mathcal{D}$ be the set of all functions on M that are differentiable at p. The **tangent vector to the curve** α at $t = 0$ is a function $\alpha'(0) : \mathcal{D} \to \mathbf{R}$ given by

$$\alpha'(0)f = \frac{d(f \circ \alpha)}{dt}\Big|_{t=0}, \qquad f \in \mathcal{D}.$$

A **tangent vector at a point** $p \in M$ is the tangent vector at $t = 0$ of some curve $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$. The set of all tangent vectors to M at p called the **tangent space** at p will be denoted T_pM. It is indeed a vector space, in fact:
If we choose a parametrization $\mathbf{x} : U \to M$ at $p = \mathbf{x}(0)$ and express $f(x^1, \dots, x^n) = f \circ \mathbf{x}(x^1, \dots, x^n)$ and $\mathbf{x}^{-1} \circ \alpha(t) = (x^1(t), \dots, x^n(t))$, we obtain

$$\alpha'(0)f = \frac{d}{dt} f(x^1(t), \dots, x^n(t))\Big|_{t=0} = V^i \frac{\partial f}{\partial x^i}, \qquad V^i = \frac{dx^i}{dt}(0)$$

so there is a one to one correspondence $V \in \mathbf{R}^n$ and tangent vectors.
We define the **tangent bundle** by $TM = \{(p, v); p \in M, v \in T_pM\}$.

A **vector field** X is a mapping $M \ni p \to X(p) \in T_pM$. It is differentiable if $X : M \to TM$ is differentiable.

Let $\varphi : M \to n$ be a differentiable mapping between differentiable manifolds. For every $p \in M$ and $v \in T_pM$ we choose a differentiable curve α with $\alpha(0) = p$, $\alpha'(0) = v$. Let $\beta = \varphi \circ \alpha$. The mapping $d\varphi_p : T_pM \to T_pN$ given by $d\varphi_p(v) = \beta'(0)$ is linear and is independent on the choice of α. It is called the **differential** of φ at p.