

Lecture 8: Riemannian manifolds.

A **Riemannian metric** on a differentiable manifold M is an inner product $\langle \cdot, \cdot \rangle_p$ (i.e. positive definite symmetric quadratic form) on the tangent spaces $T_p M$ for each $p \in M$, which varies differentiable with p in the following sense. If \mathbf{x} are local coordinates then $g_{ij} = \langle \partial/\partial x^i, \partial/\partial x^j \rangle$ are differentiable.

In new coordinates $\tilde{\mathbf{x}}$

$$\tilde{g}_{ij} = \left\langle \frac{\partial}{\partial \tilde{x}^i}, \frac{\partial}{\partial \tilde{x}^j} \right\rangle = \left\langle \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial}{\partial x^k}, \frac{\partial x^\ell}{\partial \tilde{x}^j} \frac{\partial}{\partial x^\ell} \right\rangle = \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial x^\ell}{\partial \tilde{x}^j} g_{k\ell}$$

A map $f : M \rightarrow N$ between manifolds is called a **diffeomorphism** if its differentiable, bijective with differentiable inverse. It is called an **isometry** if in addition

$$\langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)}$$

The **arc length** s along the curve $c(t)$ in M is given by

$$\int_0^t \left\langle \frac{dc}{dt}, \frac{dc}{dt} \right\rangle^{1/2} dt$$

where

$$\left\langle \frac{dc}{dt}, \frac{dc}{dt} \right\rangle = g_{ij}(x) \frac{dx^i}{dt} \frac{dx^j}{dt}.$$

and $(x_1(t), \dots, x_n(t)) \in \mathbf{R}^n$ is the inverse image of $c(t) \in M$ in the \mathbf{x} parametrization.

Ex Let M be a submanifold of dimension n of \mathbf{R}^N and let $\iota : M \rightarrow \mathbf{R}^N$ be the inclusion map. Then M has a Riemannian metric defined by restricting the Euclidean metric in \mathbf{R}^N to M ; $\langle u, v \rangle_p = \langle d\iota_p(u), d\iota_p(v) \rangle$. It is a theorem of Nash that any manifold can be isometrically embedded as a submanifold of \mathbf{R}^N if N is large.

Connections.

Let M be a differentiable manifold. Let $\mathcal{D}(M)$ be the (infinitely) differentiable functions on M and let $\mathcal{X}(M)$ denote the set of all differentiable vector fields.

An **affine connection** ∇ on M is a map $\mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$ denoted by $\nabla : (X, Y) \rightarrow \nabla_X Y$, satisfying:

- (i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$
- (ii) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$.
- (iii) $\nabla_X (fY) = f\nabla_X Y + X(f)Y$

Note the difference with Lie derivative $\mathcal{L}_X Y = XY - YX$.

Proposition Let M be a differentiable manifold with an affine connection ∇ . There exists a unique correspondence which associate to a vector field V along a curve

$c : I \rightarrow M$ another vector field DV/dt along c , called the covariant derivative of V along c , such that:

(a) $D(V + W)/dt = dV/dt + dW/dt$

(b) $D(fV)/dt = (df/dt)V + fDV/dt$

(c) If V is induced by a vector field Y ; $V(t) = Y(c(t))$, then $DV/dt = \nabla_{dc/dt}Y$.