Lecture 9: Connections.

Let M be a differentiable manifold. Let $\mathcal{D}(M)$ be the (infinitely) differentiable functions on M and let $\mathcal{X}(M)$ denote the set of all differentiable vector fields.
An **affine connection** ∇ on M is a map $\mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ denoted by $\nabla : (X, Y) \to \nabla_X Y$, satisfying:
 (i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$
 (ii) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$.
 (iii) $\nabla_X (fY) = f\nabla_X Y + X(f)\,Y$
for $f, g \in \mathcal{D}(M)$ and $X, Y, Z \in \mathcal{X}(M)$.
$\nabla_X Y$ should be thought of as the directional derivative of Y in the direction of X.
Note that the Lie derivative $\mathcal{L}_X Y = XY - YX$ does not satisfy (i).

Let $E_1, \dots, E_n$ be a local frame for TM (for now we use a coordinate frame $E_i = \partial/\partial x^i$.) Expanding in the frame

$$\nabla_{E_i} E_j = \Gamma^k_{ij} E_k \tag{1}$$

where Γ^k_{ij} are called the Christofell symbols of the connection with respect to the frame. If we expand $X = X^i E_i$, $Y = Y^j E_j$ we obtain

$$\nabla_X Y = \nabla_{X^i E_i} Y^j E_j = X^i\Big((E_i Y^j)E_j + Y^j \nabla_{E_i} E_j\Big) = \Big(XY^k + X^i Y^j \Gamma^k_{ij}\Big)E_k \tag{2}$$

It follows that $\nabla_X Y$ at $p \in M$ depends only on $X(p)$ and on Y in the direction of X.

Proposition Let M be a differentiable manifold with an affine connection ∇ and let $c : I \to M$ be a curve. Then there is a unique operator DV/dt acting on vector fields V defined along c, such that
 (a) $D(V + W)/dt = dV/dt + dW/dt$
 (b) $D(fV)/dt = (df/dt)V + fDV/dt$
 (c) If V is induced by a vector field Y; $V(t) = Y(c(t))$, then $DV/dt = \nabla_{dc/dt} Y$.
DV/dt is called the covariant derivative of V along c.
Proof Suppose that there is an operator satisfying (a), (b), (c) and expand in the frame $V = V^j E_j$ and $dc/dt = dx^i/dt E_i$. We obtain $DE_j/dt = \nabla_{dc/dt} E_j = \nabla_{dx^i/dt E_i} E_j = dx^i/dt E^k_{ij} E_k$. Hence

$$\frac{DV}{dt} = \frac{dV^j}{dt} E_j + V^j \frac{DE_j}{dt} = \Big(\frac{dV^k}{dt} + \frac{dx^i}{dt} V^j \Gamma^k_{ij}\Big) E_k \tag{3}$$

This shows uniqueness. To show existence we define DV/dt by this expression in any coordinate patch and it follows from the uniqueness already proven that its unique in the intersection of two coordinate patches.

A vector field V along a curve c is called **parallel** if $DV/dt = 0$ along c.

Proposition Let $V_0 \in T_{c(t_0}M$. Then there is a unique parallel vector field V along c such that $V(t_0) = V_0$. V is called the **parallel transport** of V_0 along c.
Pf $DV/dt = 0$ in the coordinates (3) is a system of ODEs which has a unique solution.

Riemannian Connections.

A connection is said to be **compatible** with the metric $\langle\,,\,\rangle$ when for any smooth curve c and any pair of parallel vector fields P, Q along c, we have $\langle P, Q\rangle = \text{constant}$.

Proposition A connection is compatible with a metric iff along any curve c:

$$\frac{d}{dt}\langle V, W\rangle = \left\langle \frac{DV}{dt}, W\right\rangle + \left\langle V, \frac{DW}{dt}\right\rangle \tag{4}$$

Pf It is obvious that (3) implies that the connection is compatible with the metric. Choose an orthonormal basis $\{P_1(t_0), \ldots, P_n(t_0)\}$ of $T_{c(t_0)}M$. By the previous proposition these can be parallel transported along c to $\{P_1(t), \ldots, P_n(t)\}$ and by (4) they are still orthonormal. If we write $V = V^i P_i$, $W = W^j P_j$ we have $DV/dt = dV^i/dtP_i$, $DW/dt = dW^j/dtP_j$ and

$$\left\langle \frac{DV}{dt}, W\right\rangle + \left\langle V, \frac{DW}{dt}\right\rangle = \frac{dV^i}{dt}W^i + V^i\frac{dW^i}{dt} = \frac{d}{dt}\delta_{ij}V^iW^j = \frac{d}{dt}\langle V, W\rangle.$$

Corollary A connection is compatible with the metric if and only if

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \tag{5}$$

Pf If c is a curve such that $dc/dt = X$ then (5) is equivalent to (4). The proof in the other direction is to expand in the basis in the proof of the previous proposition.