Problem set 1.

(1) Calculate the curvature κ of a curve $x(t)$ with arbitrary parametrization. Show that the tangential component of $\ddot{x}$ is equal to $\ddot{s}$ times the unit tangent vector, where $s(t)$ is the arc length, and that the normal component is $\dot{s}^2\kappa$ times the principal normal.

(2) Use Frenet formulas and the Taylor expansion for κ and τ to determine the Taylor expansion of the curve $x(s)$ with error $o(s^3)$ in terms of $x(0)$, $\dot{x}(0)$, $n(0)$, $b(0)$.

(3) Prove that if $g = \det(g_{ij})$ then $\partial_\ell g = 2g\Gamma^i_{ij}$.

(4) Show that $R_{ik} = \partial_j \Gamma^j_{ik} - \frac{1}{2}\partial_i\partial_k \ln g + \frac{1}{2}\Gamma^\ell_{ik}\partial_\ell \ln g - \Gamma^\ell_{ij}\Gamma^j_{\ell k}$

Write down a formulas for the (a) first fundamental form, (b) the second fundamental form, (c) the principal directions, (d) the principal curvatures, (e)the total curvature and (f) the differential of the Gauss map, in appropriate coordinates for the following surfaces:
(5) $z = f(x, y)$ at $(x, y) = 0$, where $f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0$.
(6) The plane.
(7) The cylinder.
(8) The sphere.
(9) The saddle $z = xy$.

(10) Let M_0 be an open subset of M which is mapped diffeomorphically into S^n (n dimensional unit sphere) by γ, and let f be a continuous function with support in the range $\gamma(M_0) \subset S^n$. Show that

$$\int f\, dS = \int (f \circ \gamma)|K| dM$$

where dS and dM are the Euclidean volume elements on S^n and M and K is the total curvature.