Lecture 10: 4.3 General Relativity.

The *equivalence principle* states that all bodies fall the same way in a gravitational field. This is not the case in an electromagnetic field which only influences charged particles. Because of this Einstein came up with the idea that instead of thinking of the gravitation as a force that accelerates bodies one should think of it as a part of spacetime. The world line of freely falling bodies in a gravitational field are simply the geodesics of a curved spacetime.

Let u be the unit 4-velocity of a particle. A free particle travels along a geodesic

$$u^a \nabla_a u^b = 0$$

where ∇ is covariant differentiation with respect to the spacetime Lorentzian metric. On the other hand a particle of mass m and charge q in an electromagnetic field F_{ab} satisfies the Lorentz force equation:

$$u^a \nabla_a u^b = \frac{q}{m} F^b{}_c u^c$$

We define the energy momentum tensor for perfect fluid is almost the same as in flat spacetime;

$$T_{ab} = \rho\, u_a u_b + P(g_{ab} + u_a u_b).$$

The divergence free condition is now

$$\nabla^a T_{ab} = 0$$

which as before leads to the equations

$$u^a \nabla_a \rho + (\rho + P)\nabla^a u_a = 0$$
$$(P + \rho) u^a \nabla_a u_b + (g_{ab} + u_a u_b)\nabla^a P = 0$$

The curved equation for the scalar field is

$$\nabla^a \nabla_a \phi - m^2 \phi = 0.$$

Here $\nabla^a \nabla_a$ is the geometric wave operator in the metric g. The energy momentum tensor

$$T_{ab} = \nabla_a \phi\, \nabla_b \phi - \frac{1}{2} g_{ab} \big(\nabla^c \phi\, \nabla_c \phi + m^2 \phi \big)$$

satisfies $\nabla^a T_{ab} = 0$. Maxwell's equations take the form

$$\nabla^a F_{ab} = -4\pi j_b$$

$$\nabla_{[a} F_{bc]} = 0 = \nabla^a * F_{ab}$$

where j^a is the current 4-vector of electric charge. The energy momentum tensor for the *electromagnetic field* is

$$T_{ab} = \frac{1}{4\pi} \big(F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{de} F^{de} \big)$$

which as before satisfy $\nabla^a T_{ab} = 0$, if $j_a = 0$.

We have now given the equations of motions in curved spacetime showing how the metric or gravity influence the motion of mass. However, we also need to give an equation for how mass influences the metric or gravitational field.

We will now explain how matter influence the gravitational field or the metric of space time. Here Einstein was influenced by some ideas going under the name of *Mach's principle* but which also go back to others like Riemann. They felt that matter should contribute to the local definition of nonaccelerating and that in a universe with no matter there should be no meaning of these concepts.

How is spacetime geometry going to be influenced by the matter distribution? In Newtonian theory the gravitational force is represented by the gradient of a gravitational potential. The gravitational potential satisfy Poisson's equation

$$\triangle\phi = 4\pi\rho$$

where ρ is the mass density. Consider two small masses m separated by a vector $\mathbf{x}$ influenced by the gravitational potential. Then the difference in force between them is $-\mathbf{x}\cdot\nabla\nabla\phi$ which is determining the relative acceleration. On the other hand the relative acceleration of two geodesics in curved space is given by the curvature $-R_{cbd}{}^{a}v^c x^b v^d$, where v^a is the 4-velocity of the particles and x is the deviation vector. This suggests the correspondence

$$R \sim \partial^2\nabla^2\phi.$$

Since $R \sim \partial^2 g$ this means that the metric is like the gravitational potential and that we should get an equation for the curvature. Also since

$$T_{ab}v^a v^b \sim \rho$$

this gives some idea of Einstein's equations

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = 8\pi T_{ab}$$