Lecture 11: 4.4 Harmonic coordinates and the linearized equations.

We will now show that Einstein's equations in the case of weak fields reduces are consistent with Newton's equations. We will assume that

$$g_{ab} = \eta_{ab} + \gamma_{ab}$$

where γ_{ab} is small compared to g_{ab}. Then modulo quadratic terms in h

$$g^{ab} = \eta^{ab} - \gamma^{ab} + O(\gamma^2), \qquad \gamma^{ab} = \eta^{ac}\eta^{bd}\gamma_{cd}$$

Two metric that differs by a diffeomorphism define the same spacetime so there is freedom of choice of representative within a diffeomorphism class. We choose to impose the harmonic coordinate condition on the metric

$$g^{ab}\Gamma_{ab}^{c} = 0,$$

where Γ_{ab}^{c} is the Christoffel symbol

$$\Gamma_{ab}^{c} = \frac{1}{2} g^{cd}\big(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\big).$$

In fact, given a metric one can always at least locally make a change of coordinates so in the new coordinates the harmonic coordinate condition hold. We just solve a system for the new coordinates $\Box_g x^d = 0$, where the geometric wave operator is

$$\Box_g \phi = g^{ab}\nabla_a \nabla_b \phi = g^{ab}\partial_a\partial_b\phi + g^{ab}\Gamma_{ab}^{c}\partial_c\phi$$

Since the geometric wave operator is invariant under changes of coordinates it must also vanish when expressed in the x^d coordinates $0 = \Box_g\, x^d = g^{ab}\partial_a\partial_b x^c + g^{ab}\Gamma_{ab}^{c}\partial_c x^d = g^{ab}\Gamma_{ab}^{d} = 0$, since $\partial_c x^d = \delta_c^d$. In the harmonic coordinates the geometric wave operator hence reduces to

$$\widetilde{\Box}_g \phi = g^{ab}\partial_a\partial_b\phi$$

Recall that

$$R_{\mu\nu\rho}{}^{\sigma} = \partial_\nu \Gamma_{\mu\rho}^{\sigma} - \partial_\mu \Gamma_{\nu\rho}^{\sigma} + \Gamma_{\mu\rho}^{\alpha}\Gamma_{\alpha\nu}^{\sigma} - \Gamma_{\nu\rho}^{\alpha}\Gamma_{\alpha\mu}^{\sigma},$$

Hence modulo terms that are quadratic in ∂h the Ricci curvature is

$$R_{\mu\rho} = R_{\mu\nu\rho}{}^{\nu} = \partial_\nu \Gamma_{\mu\rho}^{\nu} - \partial_\mu \Gamma_{\nu\rho}^{\nu} + \Gamma_{\mu\rho}^{\alpha}\Gamma_{\alpha\nu}^{\nu} - \Gamma_{\nu\rho}^{\alpha}\Gamma_{\alpha\mu}^{\nu}$$

$$= \frac{1}{2} g^{\nu d}\partial_\nu\big(\partial_\mu g_{\rho d} + \partial_\rho g_{\mu d} - \partial_d g_{\mu\rho}\big) - \frac{1}{2} g^{\nu d}\partial_\mu\big(\partial_\nu g_{\rho d} + \partial_\rho g_{\nu d} - \partial_d g_{\nu\rho}\big) + O((\partial g)^2)$$

$$= \frac{1}{2} g^{\nu d}\partial_\nu\big(\partial_\rho g_{\mu d} - \partial_d g_{\mu\rho}\big) - \frac{1}{2} g^{\nu d}\partial_\mu\big(\partial_\rho g_{\nu d} - \partial_d g_{\nu\rho}\big) + O((\partial g)^2)$$

$$= -\frac{1}{2} g^{\nu d}\partial_\nu\partial_d g_{\mu\rho} + \frac{1}{2} g^{\nu d}\partial_\rho\big(\partial_\nu g_{d\mu} + \partial_d g_{\nu\mu} - \partial_\mu g_{\nu d}\big) + \frac{1}{2} g^{\nu d}\partial_\mu\big(\partial_\nu g_{d\rho} + \partial_d g_{\nu\rho} - \partial_\rho g_{\nu d}\big) + O((\partial g)^2)$$

$$= -\frac{1}{2} g^{\nu d}\partial_\nu\partial_d g_{\mu\rho} + \frac{1}{2} g^{\nu d}\partial_\rho\big(g_{\mu c}\Gamma_{\nu d}^{c}\big) + \frac{1}{2} g^{\nu d}\partial_\mu\big(g_{\rho c}\Gamma_{\nu d}^{c}\big) + O((\partial g)^2)$$

If the metric satisfy the harmonic coordinate condition $g^{\nu d}\Gamma_{\nu d}^{c} = 0$ then

$$R_{\mu\rho} = -\frac{1}{2}\widetilde{\Box}_g\, \gamma_{\mu\rho} + O(\partial\gamma)^2$$

Moreover we have

$$g_{\mu\rho}R = g_{\mu\rho}g^{\alpha\beta}R_{\alpha\beta} = -\frac{1}{2}\eta_{\mu\rho}\widetilde{\Box}_g(\eta^{\alpha\beta}\gamma_{\alpha\beta}) + O(\gamma\partial^2\gamma) + O(\partial\gamma)^2$$

and hence with $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$:

$$G_{\mu\rho} = -\frac{1}{2}\Box\,\overline{\gamma}_{\mu\rho} + O(\gamma\partial^2\gamma) + O(\partial\gamma)^2, \qquad \text{where} \quad \overline{\gamma}_{\mu\rho} = \gamma_{\mu\rho} - \frac{1}{2}\eta_{\mu\rho}\eta^{\alpha\beta}\gamma_{\alpha\beta}.$$

4.4a The Newtonian Limit.

When gravity is weak the linearized version of Einstein's equations should be valid

$$\Box\overline{\gamma}_{ab} = -16\pi T_{ab}, \qquad \Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$$

Our assumption on the sources is then

$$T_{ab} = \rho\, t_a t_b,$$

where $t = (1, 0, 0, 0)$ is the time direction. (The neglect of time-space component is essentially the statement that the velocity is small and the neglect of space-space component is the statement that stresses are small.) We also assume that the sources are slowly varying so we also expect the time derivatives of $\overline{\gamma}_{ab}$ to be small. In that case the equation become

$$\triangle\overline{\gamma}_{\mu\nu} = 0, \quad (\mu, \nu) \neq (0, 0), \qquad \triangle\overline{\gamma}_{00} = -16\pi\rho$$

where $\triangle = \sum_{i=1}^{3} \partial_i^2$ is the space Laplacian. Hence a solution is given by

$$\gamma_{ab} = \overline{\gamma}_{ab} - \frac{1}{2}\eta_{ab}\eta^{\alpha\beta}\gamma_{\alpha\beta} = -(4t_a t_b + 2\eta_{ab})\phi$$

where $\phi = -\overline{\gamma}_{00}/4$ is a solution of Poisson's equation

$$\triangle\phi = 4\phi\rho$$

The motion of test bodies in curved spacetime is governed by the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0,$$

For a motion of a particle much slower than the speed of light we may approximate $dx^a/d\tau$ by $(1, 0, 0, 0)$ in the right and we get

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{00} = \frac{1}{2}\frac{\partial\gamma_{00}}{\partial x^\mu} = -\frac{\partial\phi}{\partial x^\mu}$$

where again the time derivatives have been neglected. Thus the motion of test bodies have acceleration

$$a = -\nabla\phi$$

which is of course Newton's equation in the field of a gravitational potential ϕ.