Lecture 13: 6.1 The Schwarzschild spacetime.

A spacetime is called static if it can be written in the form

$$ds^2 = -V^2(x^1, x^2, x^3)dt^2 + \sum_{\mu,\nu=1}^{3} h_{\mu\nu}(x^1, x^2, x^3)\, dx^\mu dx^\nu$$

The only spherically symmetric static solution, in fact the only spherically symmetric solution, to Einstein's vacuum equations is the *Schwarzschild spacetime*:

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the metric on the unit sphere. Here r is not exactly the distance to a center, but rather it should be interpreted as $r = (A/4\phi)^{1/2}$, where A is the area of the sphere with fixed r. One can check that the expression above satisfy Einstein's equations $R_{\mu\nu} = 0$ expressed in the coordinates (t, r, θ, ϕ).

This solution is to be thought of as a relativistic version of the Newtonian potential field around a star or planet M/r. The expression above is singular when $r = 2M$ and $r = 0$ so in a sense its really two spacetimes one exterior when $r > 2M$ and one interior where $0 < r < 2M$. For the sun the radius at $r = 2M$ is far inside the sun and only the part outside the sun can be modeled by the vacuum equations and it has to be matched to a smooth solution of the Einstein's equations with a matter field in the interior. The singularities at $r = 0$ and $r = 2M$ are relevant only for bodies which have undergone a complete gravitational collapse. A black hole is modeled by the interior region.

6.4 The Kruskal coordinates and black holes.

The Schwarzschild metric expressed in the coordinates (t, r, θ, ϕ) have singularities at $r = 0$ and $r = 2M$, but are these singularities real or could it be that in some other coordinate system there are no singularities. The first test of this is to form quantities from the curvature that are independent of coordinates, such as $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$. It turns out that this quantity blows up when $r \to 0$ and hence excluding the possibility that the metric would be regular there in some other coordinate system. This is not the case for $r = 2M$ and as we will see one can find other coordinates in which the metric is regular at $r = 2M$. The situation of a coordinate singularity is easy to illustrate with an example. The metric

$$ds^2 = -dt^2/t^4 + dx^2$$

appear to be singular when $t = 0$. However a change of variables $\tau = 1/t$ gives that

$$ds^2 = -d\tau^2 + dx^2$$

To find a coordinate system that will resolve the singularity when $r = 2M$ in the expression for the Schwarzschild metric it is natural to introduce characteristic coordinates. The characteristics that do not depend on the angles are given by

$$t = \pm r_* + t_0, \qquad \text{where} \qquad r_* = r + 2M \ln\Big|\frac{r}{2M} - 1\Big|.$$

It can easily be checked that this is a characteristic curve by calculating the tangent vector and evaluate the metric on it to see that it is a null geodesic. The characteristic coordinates are given by

$$u = t - r_*, \qquad v = t + r_*$$

In these coordinates the Schwarzschild metric become

$$ds^2 = -(1 - 2M/r)dudv + r^2 d\Omega^2$$

There is no longer a singularity at $r = 2M$ but the metric is still degenerate there. We can further introduce $U = e^{-u/4M}$ and $V = e^{v/4M}$ to obtain the metric

$$ds^2 = -\frac{32M^3 e^{-r/2M}}{r} dUdV + r^2 d\Omega^2$$

and finally with $T = (U + V)/2$ and $X = (V - U)/2$ we obtain

$$ds^2 = \frac{32M^3 e^{-r/2M}}{r}(-dT^2 + dX^2) + r^2 d\Omega^2$$

In this coordinates $r = 2M$ corresponds to $T^2 = X^2$ where there no longer is a singularity. The singular set $r = 0$ corresponds to $T^2 - X^2 = 1$. The solution can hence be extended to the region where $T^2 - X^2 < 1$ or $r > 0$. However, something peculiar still happens at $r = 2M$ in the original coordinates the two families of characteristics can only approach the set where $r = 2M$ from $r > 2M$ in infinite t variable. However, in the Kruskal coordinates we can approach the set where $T^2 - X^2$ along a geodesic, which in these coordinates are just straight lines $T = \pm X + T_0$, for finite T. Since t or T does not have an interpretation as time we must measure the proper time along the incoming characteristics and that is finite since it is in the nonsingular Kruskal coordinates. The conclusion is therefore that in positive time you can get in along a null geodesic to the region where $T^2 - X^2 > 0$ but you can never get out of this region. The region where $0 < T^2 - X^2 < 1$ is called a black hole since light can not get out from it.