Lecture 15: 7.1 Axial symmetry.

We will here consider spacetimes which are stationary and axisymmetric. This is of great interest since it describes the gravitational field away from a spherical rotating star. In Newtonian physics the field is independent of the rotation but in general relativity it is not.

The A spacetime is called *stationary* if there is a one parameter group of isometries σ_t whose orbits are timelike curves, i.e. $\sigma_{t+s} = \sigma_t \circ \sigma_s$, $\sigma_t^* g = g$ and $g_{ab}\dot{\sigma}_t^a \dot{\sigma}_t^b < 0$. The generator $\xi = \dot{\sigma}_0$ is a timelike killing vector field $\mathcal{L}_\xi g_{ab} = \lim_{t\to 0} \frac{1}{t}\big(\phi_t^* g_{ab} - g_{ab}\big) = 0$ or $\nabla_a \xi_b + \nabla_b \xi_a = 0$.
A spacetime is called *axisymmetric* if there is a one parameter group of isometries χ_ϕ whose orbits are spacelike curves. The generator $\psi = \dot{\chi}_0$ is a spacelike killing vector field $\mathcal{L}_\psi g_{ab} = 0$ or $\nabla_a \psi_b + \nabla_b \psi_a = 0$.
We will assume that we have a stationary and axisymmetric spacetime such that the time translations and rotations commute $\sigma_t \circ \chi_\phi = \chi_\phi \circ \sigma_t$, which is seen to be equivalent to that the generators commute $[\xi, \psi] = 0$. If the generators commute we can choose coordinates $x^0 = t$, $x^1 = \phi$,x^2, x^3 such that $\partial_t = \xi$ and $\partial_\phi = \psi$. Since $\mathcal{L}_{\partial_t} g_{ab} = \partial_t g_{ab} = 0$ and $\mathcal{L}_{\partial_\phi} g_{ab} = \partial_\phi g_{ab} = 0$ it follows that a stationary axisymmetric metric can be written

$$ds^2 = g_{\mu\nu}(x^2, x^3)\, dx^\mu\, dx^\nu$$

Under two further assumptions that (i) $\xi_{[a}\psi_b \nabla_c \xi_{d]}$ and $\xi_{[a}\psi_b \nabla_c \psi_{d]}$ each vanish at at least one point, which is true for rotations which vanish on the axis of symmetry, and (ii) $\xi^a R_a{}^{[b}\xi^c\psi^{d]} = \psi^a R_a{}^{[b}\xi^c\psi^{d]} = 0$ which is true for solutions of the Einstein vacuum equation's, the spacetime can be written

$$ds^2 = -V(dt - wd\phi)^2 + V^{-1}\rho^2 d\phi^2 + \Omega^2(d\rho^2 + \Lambda dz^2)$$

where (ρ, z) are to be thought of as cylindrical coordinates.

12.3 The Kerr metric.

The only stationary axisymmetric solution to the Einstein vacuum equations is the *Kerr metric*:

$$ds^2 = -\frac{\Lambda - a^2\sin^2\theta}{\Sigma}dt^2 - \frac{2a\sin^2\theta\big(r^2 + a^2 - \Lambda\big)}{\Sigma}dtd\phi + \frac{(r^2+a^2)^2 - \Lambda a^2\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 + \frac{\Sigma}{\Lambda}dr^2 + \Sigma d\theta^2$$

where

$$\Lambda = r^2 + a^2\cos^2\theta, \qquad \Lambda = r^2 + a^2 - 2Mr$$