lecture 17: 8.1.

We say that $p \in M$ is a *future endpoint* of a curve $\lambda(t)$ if for every neighborhood O of p there exists a t_0 such that $\lambda(t) \in O$ for all $t > t_0$. A curve is said to be *future inextendible* if it has no future endpoint. Past inextendible is defined similarly.
Lemma Let λ be a past inextendible causal curve passing through point p.
Then through any $q \in I^+(p)$ there exists a past inextendible timelike curve $\gamma \in I^+(\lambda)$.

Let $\{\lambda_n\}$ be a sequence of causal curves. A point $p \in M$ is called a *convergence point* if for every neighborhood O of p, there is an N such that $\lambda_n \cap O \neq \emptyset$ for $n \geq N$. Its called a *limit point* if the same is true for a subsequence. A curve λ is said to be a *convergence/limit curve* for $\{\lambda_n\}$ if each $p \in \lambda$ is a convergence/limit point.
Lemma A sequence of future inextendible causal curves $\{\lambda_n\}$, with a limit point p, has a future inextendible causal limit curve λ passing through p.

8.2 Causality conditions.

In order for a spacetime to be causally wellbehaved we do not want it to exist any closed causal curves or even any curves that are close to coming back to themselves since then any small perturbation of the metric could produce causality violation.

We say that a spacetime is *strongly causal* if for all $p \in M$ and every neighborhood O of p, there is a neighborhood V of p contained in O such that no causal curve intersects V more than once. (This means that it enters and or exists V at most once, i.e. the set of parameter values for which the curve is in V is connected.) This prevents a curve from being close to coming back but other things could happen.
Lemma Let (m, g_{ab}) be strongly causal and let $K \subset M$ be compact. Then every causal curve confined within K must have past and future endpoints in K.

We say that a spacetime (M, g_{ab}) is *stable causal* if there a timlike vector field t^a such that the spacetime $(M, \tilde{g}_{ab})$ has no closed timelike orbits, where $\tilde{g}_{ab} = g_{ab} - t_a t_b$. The light cones for $\tilde{g}_{ab}$ are strictly larger than the light cones for g_{ab}, i.e. every casual vector for g_{ab} is a timelike vector for $\tilde{g}_{ab}$. Stable causal is equivalent to the existence of a global time function:
Theorem A spacetime is stable causal if and only if there a global differentiable function f such such that $\nabla^a f$ is a past directed timelike vector.
Proof Since $\nabla^c f$ is past directed timelike, along every future directed timelike curve with tangent v^a, we have $v(f) = g_{ab} v^a \nabla^b f > 0$. (This follows since if (t, x) and (s, y) are future directed timelike vectors in Minkowski space then $(t, x) \cdot (s, y) = -ts + x \cdot y < -|x|\,|y| + x \cdot y \leq 0$.) It follows that f is increasing along any timelike curve so there can not be any closed timelike curves. The same thing can be showbn to be true for f in the metric $\tilde{g}_{ab}$.
Corollary Stable causal implies strong causal.
Proof Since f increases along and future directed causal curve f can not enter the region twice.