Lecture 18: 8.3 Domain of Dependence; Global Hyperbolicity.

A subset S is called *achronal* if no points in S can be joined by a timelike curve. For S closed and achronal, we define the *edge* of S as the set of points $p \in S$ such that every open neighborhood O of p contains $q \in I^+(p)$ and $r \in I^-(p)$ and a timelike curve from r to q which does not intersect S.
Theorem A closed achronal set without edge (*a slice*) is a 3-dim C^0 submanifold.

The *future domain of dependence* $D^+(S)$, of a closed achronal set S, is the set of points $p \in M$ such that every past inextendible causal curve through p intersects S. What happens in $D^+(S)$ should be determined by the initial conditions on S.
The *past domain of dependence* $D^-(S)$ is defined similarly and
the *domain of dependence* is $D(S) = D^+(S) \cup D^-(S)$.

A closed achronal set Σ for which $D(\Sigma) = M$ is called a *Cauchy (hyper)surface*.
It is in fact a hypersurface by the previous theorem.
A spacetime which has a Cauchy hypersurface is said to be *globally hyperbolic*.

For a closed achronal set S the *future Cauchy horizon* is defined to be
$H^+(S) = \overline{D^+(S)} \setminus I^-(D^+(S))$.
Theorem Every $p \in H^+(S)$ lies on a null geodesic λ contained entirely within $H^+(S)$ which either is past inextendible or has a past endpoint on the edge of S.
$H^-(S)$ is defined similarly and the *Cauchy horizon* is $H(S) = H^+(S) \cup H^-(S)$.
Proposition $H(S) = \dot{D}(S)$, the boundary. Moreover, if M is connected then a closed achronal subset Σ is a Cauchy surface if and only of $H(\Sigma)$ is empty.

Theorem If Σ is closed, achronal, edgeless subset, then Σ is a Cauchy surface if and only if every inextendible null geodesic intersects Σ and enters $I^+(\Sigma)$ and $I^-(\Sigma)$.

Let $C(p,q)$ denote the set of continuous future directed causal curves from p to q. We define a topology on $C(p,q)$ the open sets consist of curves contained open sets.
Theorem Let (M, g_{ab}) be globally hyperbolic. Then $C(p,q)$ is compact.

Theorem Let (M, g_{ab}) be a globally hyperbolic spacetime. Then it is stable causal. A global time function can be chosen such that its level sets Cauchy hypersurfaces.