

Lecture 2: 2.1 Manifolds.

A *topological manifold* M of dimension n is a Hausdorff topological space with a countable basis of open sets that locally looks like \mathbf{R}^n in the following sense each $p \in M$ has a neighborhood which is homeomorphic to an open subset of \mathbf{R}^n . A homeomorphism is a continuous invertible function with continuous inverse.

The pair (U, ϕ) where O is an open subset of M and $\phi : O \rightarrow U$ is a homeomorphism onto an open subset U of \mathbf{R}^n is called a coordinate neighborhood.

A C^∞ manifold M of dimension n is a topological manifold with a family of coordinate charts $\{(O_\alpha, \phi_\alpha)\}$ such that

- (i) $\cup_\alpha O_\alpha = M$.
- (ii) If $O_\alpha \cap O_\beta \neq \emptyset$ then $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(O_\alpha \cap O_\beta) \rightarrow \phi_\alpha(O_\alpha \cap O_\beta) \rightarrow$ is a C^∞ map.

Ex \mathbf{R}^n is a manifold, in fact with $O = \mathbf{R}^n$ and the identity map.

Ex The sphere $S^2 = \{(x^1, x^2, x^3) \in \mathbf{R}^3; (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$ is a manifold. In fact, the coordinate charts are the open sets $O_i^\pm = \{(x^1, x^2, x^3) \in S^2; \pm x^i > 0\}$ and the maps are the projections to the unit discs in the coordinate planes.

Let M and M' be C^∞ manifolds with charts $\{(O_\alpha, \phi_\alpha)\}$ and $\{(O'_\alpha, \phi'_\alpha)\}$ respectively.

Then the product $M \times M'$ is a C^∞ manifold with charts $\{(O_\alpha \times O'_\beta, (\phi_\alpha, \phi'_\beta))\}$.

We say that a map $f : M \rightarrow M'$ is C^∞ if $\phi'_\beta \circ f \circ \phi_\alpha^{-1} : U_\alpha \rightarrow U'_\beta$ is C^∞ for any α and β . If f is also one-to-one and onto with C^∞ inverse we say that f is a diffeomorphism, and that M and M' are diffeomorphic.

2.2 Tangent vectors.

Euclidean space has a natural vector space structure, we can e.g. add forces to get the total force acting on a particle. There is no such addition on a manifold. Instead we hope to be able to add infinitesimal displacement at a point.

For a manifold embedded in Euclidean space there is a natural notion of tangent vectors and tangent space. However, even though all manifolds can be embedded in Euclidean space there might not be a natural embedding so we would like to have a more intrinsic definition. We will use that there is a one to one correspondence between vectors and directional derivatives in \mathbf{R}^n : $(v^1, \dots, v^n) \rightarrow v^\mu \partial_\mu$.

We define a tangent vector at a point $p \in M$ in a manifold M to be a map $C^\infty(M, \mathbf{R}) \rightarrow \mathbf{R}$ which is linear and satisfies Leibnitz rule:

- (i) $v(af + bg) = av(f) + bv(g)$
- (ii) $v(fg) = f(p)v(g) + g(p)v(f)$

It is clear from the definition that the space of tangent vectors at a point V_p is a vector space. In fact its an n dimensional vector space:

Th If M is an n dimensional smooth manifold and $p \in M$ then $\dim V_p = n$.

In fact let (U, ϕ) be a chart containing p and define $X_\mu(f) = \frac{\partial}{\partial x^\mu}(f \circ \phi^{-1}) \Big|_{\phi(p)}$,

where (x^1, \dots, x^n) are the coordinates in $\phi(O) \subset \mathbf{R}^n$.

Then X_μ , $\mu = 1, \dots, n$, are linearly independent since $X_\mu(x^\nu \circ \phi) = \delta_\mu^\nu$.

Using the Taylor expansion of f one can show that $v(f) = v^\mu X_\mu(f)$, $v^\mu = v(x^\mu \circ \phi)$.