Lecture 3: 2.2 Tangent vectors.

Suppose that we have a manifold M and we have to different systems of coordinates. Then we have two possible expressions for a tangent vector:

$$v = v^\mu \frac{\partial}{\partial x^\mu} = v'^\nu \frac{\partial}{\partial x'^\nu}$$

The change of variable going from $x \to x'$ gives

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu}$$

Substituting this gives

$$v'^\nu = v^\mu \frac{\partial x'^\nu}{\partial x^\mu}$$

There is an alternative characterization of tangent space at a point $p \in M$ as an equivalence class of smooth curves going through p that have the same tangent vector at p in any coordinate system. A smooth curve through $p \in M$ is a smooth map from an interval $I \ni 0$ to the manifold $C : I \to M$, such that $C(0) = p$. The curve defines a tangent vector by for any $f \in C^\infty(M, \mathbf{R})$ in local coordinates $x(t) = \phi \circ C(t)$;

$$T(f) = \frac{d}{dt} f \circ C(t)\Big|_{t=0} = \frac{dx^\mu}{dt}\Big|_{t=0} \frac{\partial}{\partial x^\mu}\big(f \circ \phi^{-1}\big)$$

We have so far defined the space of tangent vector at a point V_p. A smooth vector field is an assignment of an tangent vector $v\big|_p \in V_p$ at each point $p \in M$, that vary smoothly with p in the sense that $v(f)$ is a smooth function of p for each smooth f. This is equivalent to that the component functions are v^μ are smooth.

A *one-parameter group of diffeomorphisms* ϕ_t is a map $I \times M \to M$ such that $\phi_t : M \to M$ is a diffeomorphism, for each $t \in I$ and $\phi_{t+s} = \phi_t \circ \phi_s$. The tangent vector to the curve $t \to \phi_t$ at $t = 0$ is a vector field. Conversely, given a vector field its integral curves are solutions of the system of ordinary differential equations in local coordinates

$$\frac{dx^\mu}{dt} = v^\mu(x^1, \dots, x^n)$$

2.3 Tensors and the cotangent space. An example of tensor in physics is the stress tensor defined as follows. Consider a small plane segment with unit normal $\mathbf{n}$. Let $\mathbf{F}$ be the force that a small mass on one side of the plane asserts on an equal mass on the other side of the plane and let ℓ be another unit vector. The stress tensor $T(\mathbf{n}, \ell) = T_{ij} n^i \ell^j$ is the component of the force $\mathbf{F}$ in the ℓ direction.

Let V be a vector space of dimension n and let V^* be the *dual vectors space*, of linear maps $f : V \to \mathbf{R}$. If $v_1, \dots, v_n$ is a basis for V the we can find a dual basis $v^{*1}, \dots, v^{*n}$ for V^* such that $v^{*\mu}(v_\nu) = \delta^\mu_\nu$. (Here $\delta^\nu_\mu = 1$, if $\mu = \nu$ and $\delta^\mu_\nu = 0$ if $\mu \neq \nu$.) The correspondence $v_\mu \to v^{*\mu}$ gives an isomorphism between V and V^*, but this isomorphism depends on the choice of basis $\{v_\mu\}$ so there is no natural way of identifying V and V^* unless we have a metric. The dual of the dual V^{**} can however naturally be identified with V. In fact if $v \in V$ and $w^* \in V^*$ then $v(w^*) = w^*(v)$ defines an element of V^{**} and this map must be onto since the dimension of V is the same as that of V^{**}.

A *tensor, T, of type (k,ℓ) over V*, is a multilinear map

$$T : V^* \times \cdots \times V^* \times V \times \cdots \times V \to \mathbf{R}$$

of k copies of V^* and ℓ copies of V.

A contraction of a tensor T is the tensor $CT = T(\ldots, v^{*\mu}, \ldots; \ldots, v_\mu, \ldots)$. This can be seen to be independent of basis just like the trace of a matrix is.

Given a tensor T of type (k,ℓ) and a tensor T' of type (k',ℓ') we can construct a new tensor of type $(k+k', \ell+\ell')$ called the outer product $T \otimes T'$ by

$$\begin{aligned} &T \otimes T'\big(v^{*1}, \ldots, v^{*k+k'}, v_1, \ldots, v_{\ell+\ell'}\big) \\ &\qquad\qquad = T\big(v^{*1}, \ldots, v^{*k}, v_1, \ldots, v_\ell\big)\; T\big(v^{*k+1}, \ldots, v^{*k+k'}, v_{\ell+1}, \ldots, v_{\ell+\ell'}\big) \end{aligned}$$

Given a basis $v_1, \ldots, v_n$ of V and a dual basis $v^{*1}, \ldots v^{*n}$ of V^* we can express a general tensor T of type (k,ℓ) as

$$T = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_\ell}\; v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes v^{*\nu_1} \otimes \cdots \otimes v^{*\nu_\ell}.$$

We are in particular interested in the case $V = V_p$ is the tangent space at a point p of a manifold M. In that case we can choose the basis for V to be the vectors $\partial/\partial x^1, \ldots, \partial/\partial x^n$ in some local coordinates. It is then customary to denote the dual basis by $dx^1, \ldots, dx^n$, i.e. dx^μ is the linear transformation on the tangent space such that $dx^\mu(\partial/\partial x^\nu) = \delta^\mu_\nu$. There are also a couple of other ways to interpret the notation dx^μ. If $f : M \to \mathbf{R}$, the differential df at p is the linear map $V_p \to \mathbf{R}_p = \mathbf{R}$ of tangent vectors given by $df(v) = v(f)$. Hence dx^μ denotes the differential of the coordinate function x^μ we have $dx^\mu(\partial/\partial x^\nu) = \partial/\partial x^\nu(x^\mu) = \delta^\mu_\nu$.

There is however, an alternative way of describing the cotangent space V_p^* as equivalence classes of functions $f : M \to \mathbf{R}$ vanishing at p modulo functions vanishing to higher order or products of functions vanishing at p. Then df denotes the equivalence class of f. This is completely analogous to the description of a tangent vector $v = C'(t)$ as an equivalence classes of curves $C : \mathbf{R} \to M$. It is clear that the pairing $df(v) = v(f)$ is independent of representative

The reason we need to consider both tangent and cotangent vectors is that both show up naturally. The tangent vectors are the derivatives of functions to the manifolds such as a path of a particle and cotangent vectors are the derivatives of functions on the manifold such as the differential of the temperature. The paring between the velocity vector to the particle and the temperature differential is then the change of temperature the particle feels.

The reason we need to differentiate between tangent and cotangent vectors is that they transform differently under changes of coordinates. Using the pairing we see that the components of a cotangent vector transform like

$$\omega'_{\mu'} = \omega_\mu \frac{\partial x^\mu}{\partial x^{\mu'}}$$

As for vectors its easy to check that the components of a tensor T of type (k,ℓ) transform like

$$T'^{\mu'_1 \cdots \mu'_k}{}_{\nu'_1 \cdots \nu'_\ell} = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell} \frac{\partial x'^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\mu'_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x'^{\mu'_1}} \cdots \frac{\partial x^{\nu_\ell}}{\partial x'^{\mu'_\ell}}$$