Lecture 6: 3.4a How to calculate the curvature.
Recall that the curvature was defined by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\,\omega_c = R_{abc}{}^d\omega_d.$$

We will now derive a formula for the curvature in terms of the metric. We have

$$\nabla_b\omega_c = \partial_b\omega_c - \Gamma_{bc}^d\omega_d$$

and

$$\nabla_a\nabla_b\omega_c = \partial_a\big(\partial_b\omega_c - \Gamma_{bc}^d\omega_d\big) - \Gamma_{ab}^e\big(\partial_e\omega_c - \Gamma_{ec}^d\omega_d\big) - \Gamma_{ac}^e\big(\partial_b\omega_e - \Gamma_{be}^d\omega_d\big)$$

using the symmetry $\Gamma_{ab}^d = \Gamma_{ba}^d$ and the commutativity of partial derivatives we get

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\,\omega_c = -\partial_a\Gamma_{bc}^d\omega_d + \partial_b\Gamma_{ac}^d\omega_d + \Gamma_{ac}^e\Gamma_{be}^d\omega_d - \Gamma_{bc}^e\Gamma_{ae}^d\omega_d$$

and hence

$$R_{abc}{}^d = -\partial_a\Gamma_{bc}^d + \partial_b\Gamma_{ac}^d + \Gamma_{ac}^e\Gamma_{be}^d - \Gamma_{bc}^e\Gamma_{ae}^d$$

Using that $\Gamma_{ab}^d = g^{dc}\big(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}\big)/2$ we see that $R_{abc}{}^d \sim \partial^2 g + (\partial g)^2$.

3.2 cont. The Ricci curvature and the Einstein tensor.
If we trace the Riemann curvature tensor $R_{abc}{}^d$ we get the Ricci curvature

$$R_{ac} = R_{adc}{}^d$$

and if we trace again the scalar curvature

$$R = R_{ac}g^{ac}$$

Einstein's vacuum equations without any forces from matter are

$$R_{ab} = 0.$$

We will motivate these later but for now let us just say that if we want an equation for the metric that is invariant under changes of coordinates, under changes of accelerating frame it has to be in terms of the curvature. Moreover since we expect physics to be such that a particles path in the absence of exterior forces is determined by its initial position and velocity, then it has to be a second order equation. Therefore it has to be an equation in terms of the curvature itself. If the Riemann curvature vanishes then the metric can be transformed to the flat Minkowski metric by a change of coordinates. Just saying that the scalar curvature vanishes is too restrictive so what is left is to say that the Ricci curvature vanishes.

In the presence of exterior forces the above equations have to be modified since since its not automatically divergence free. The Einstein tensor is defined by

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$$

It follows from the Biachi identity that

$$\nabla^a G_{ab} = 0, \qquad T^a = g^{ac}\nabla_c$$

Einstein's equations in the presence of exterior forces is

$$G_{ab} = T_{ab}$$

where T_{ab} is the energy momentum tensor of the matter fields that is expect to satisfy additional equations, in particular it has to be divergence free $\nabla^a T_{ab} = 0$.

The trace free part called the Weyl tensor C_{abcd} is defined by

$$R_{abcd} = C_{abcd} + \frac{2}{n-2}\big(g_{a[c}R_{db]} - g_{b[c}R_{d]a}\big) - \frac{2}{(n-1)(n-2)}Rg_{a[c}g_{d]b}.$$

3.3 Geodesics.

A geodesic is the straightest curve between two points. In flat space this is a line but on the sphere it is great circles. In Riemannian geometry with a positive definite metric g_{ab} the curve between two points with the shortest length is a geodesic. The length of a curve $x(t)$ is defined by

$$\ell(x) = \int_a^b \sqrt{g_{ab}(x) \frac{dx^a}{dt} \frac{dx^b}{dt}}\, dt$$

Consider a family of curves $x(t,s)$ depending on an additional parameter s with the same starting and ending points and let $\delta x = \partial_s x$. Then after differentiating and integrating by parts we get

$$\delta\ell(x) = -\int_a^b \Big(\frac{d^2 x^c}{dt^2} + \Gamma_{ab}^c \frac{dx^a}{dt} \frac{dx^b}{dt} \Big) g_{cd}\, \delta x^d\, dt$$

In order for the length to be minimal we must have that $\delta\ell = 0$ for all δx which is true if

$$\frac{d^2 x^c}{dt^2} + \Gamma_{ab}^c \frac{dx^a}{dt} \frac{dx^b}{dt} = 0$$

This system of differential equations have a unique solution for any initial position and velocity. This means that for each tangent vector T at a point p we have a geodesic. We can hence define a map from the tangent space to to the geodesic at parameter value one in the manifold. This is called geodesic normal coordinates.

This is the equation for a *geodesic* and it can also be written

$$T^a \nabla_a T^b = 0, \qquad T^a = \frac{dx^a}{dt}$$

In Lorentzian geometry with a metric with signature $(-1,1,1,1)$ this equation still makes sense even though a geodesic no longer has the interpretation as the minimal distance between two points in general. For a curve that is space like, i.e. $g_{ab}T^aT^b > 0$, it still has this interpretation. However for a curve that is timelike $g_{ab}T^aT^b < 0$ we define the *proper time* to be

$$\ell(x) = \int_a^b \sqrt{-g_{ab}(x) \frac{dx^a}{dt} \frac{dx^b}{dt}}\, dt$$