The motion of the free surface of a liquid

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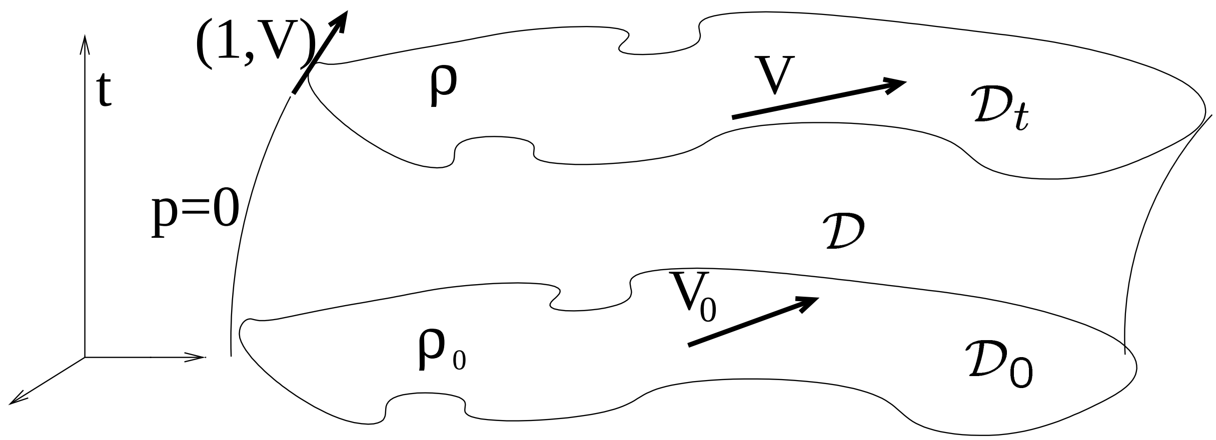
Motion of a liquid body in vacuum
(the ocean or a star)
Incompressible or compressible perfect fluid
Without surface tension and gravitation
v-velocity, p-pressure, ρ-density, t-time



Free boundary problem:
The velocity tells the boundary where to move.
The boundary is the zero set of the pressure and the pressure determines the acceleration.
(Regularity of the boundary is intimately connected to the regularity of the velocity.)

Euler's equations

$$\rho\,(\partial_t + V^k\partial_k)v_i = -\partial_i p \quad \text{in } \mathcal{D} \quad i{=}1,...,n \quad (1)$$

$$(\partial_t + V^k\partial_k)\rho + \rho\,\text{div}\,V = 0, \quad \text{in } \mathcal{D} \quad (2)$$

$\partial_k{=}\frac{\partial}{\partial x^k}$, $V^k{=}v_k$, $V^k\partial_k{=}\sum_{k=1}^n V^k\partial_k$, $\text{div}V{=}\partial_k V^k$

Equation of state

$$\text{Compressible case: } p = p(\rho), \quad (3)$$

$$p(\bar\rho_0) = 0,\ \bar\rho_0 > 0, \quad p'(\rho) > 0,\ \rho \geq \bar\rho_0 \quad (4)$$

$$\text{Incompressible case: } \rho = \text{constant} \quad (5)$$

Boundary conditions

$$(\partial_t + V^k\partial_k)|_{\partial\mathcal{D}} \ \in T(\partial\mathcal{D}) \quad (6)$$

$$p = 0, \qquad \text{on } \partial\mathcal{D} \quad (7)$$

$T(\partial\mathcal{D})$ is the tangent space of the boundary.

Initial conditions

$$\{x; (0,x) \in \mathcal{D}\} = \mathcal{D}_0 \quad (8)$$

$$V(0,x) = V_0(x),\ \rho(0,x) = \rho_0(x),\ \text{in } \mathcal{D}_0 \quad (9)$$

Compatibility cond. Formal power series solution $(\tilde V, \tilde\rho)$ in time should satisfy bound. cond.

$$(\partial_t{+}\tilde V^k\partial_k)^j(\tilde\rho{-}\bar\rho_0)\Big|_{\{0\}\times\partial\mathcal{D}_0}{=}0,\ j{=}0,... (10)$$

Local Existence?:
Given a domain $\mathcal{D}_0 \subset \mathbf{R}^\mathbf{n}$, a vector field V_0 and a function ρ_0 in $\mathcal{D}_0$ satisfying the compatibility conditions (10). Find a domain $\mathcal{D} = \cup_{0\leq t\leq T}\{t\} \times \mathcal{D}_t$, $\mathcal{D}_t \subset \mathbf{R}^\mathbf{n}$, a vector field V and a function ρ in $\mathcal{D}$, such that (1)-(9) hold.

Local existence for analytic data
Baouendi-Goulaouic, Nishida
(incompressible irrotational case)

Instability in Sobolev norms?
Rayleigh-Taylor Instability
(heavier fluid above lighter)
Ebin's counterexample (when $p<0$, $\nabla_N p>0$).

Physical condition

$$\nabla_N p \leq -c_0 < 0, \quad \text{on } \partial\mathcal{D}_0, \tag{11}$$

where $\nabla_N = N^k\partial_k$ and N is the exterior normal
Since the pressure of a fluid has to be positive
Needed for local existence in Sobolev Spaces.

Vorticity: $\operatorname{curl} v_{ij} = \partial_i v_j - \partial_j v_i$
Incompressible fluid: $\operatorname{div} V = 0$
Irrotational fluid: $\operatorname{curl} v = 0$.

Local existence in Sobolev spaces:

I) Incompressible Irrotational case:

Local existence for Water wave problem:
Yosihara, Nalimov: close to still water in $\mathbf{R}^2$
Wu: in general in $\mathbf{R}^2$ and $\mathbf{R}^3$
(no instability when water wave turns over, physical cond. hold in the irrotational case)

II) General Incompressible case:

Ebin: local exist with surface tension(announced)
Christodoulou-L: i) Sobolev norms remain bounded as long as the physical cond. hold, first order derivatives of the velocity and the second fundamental form of the free surface are bounded.
ii) local *a priori* bounds for Sobolev norms.
L: iii) Local existence assuming physical cond.

III) General Compressible case:

L: Local existence assuming physical cond.

IV) Generalizations:

L: Newtonian self gravity, special relativity.
General Relativity: Existence in special cases by Rendall, Christodoulou, Friedrich.

Irrotational Incompressible case

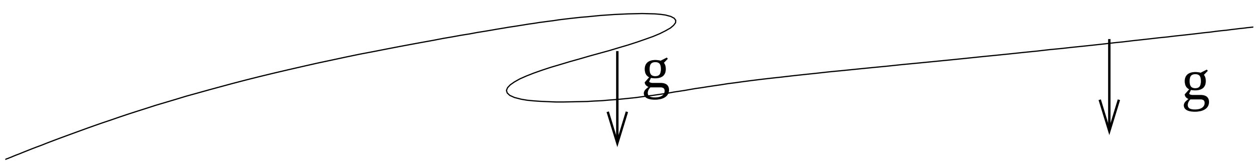
$$(\partial_t + V^k \partial_k) v_i = -\partial_i p \qquad (12)$$

$$\operatorname{div} V = 0, \qquad \operatorname{curl} v = 0 \qquad (13)$$

Taking the divergence of (12) using (13):

$$\triangle p = -(\partial_i V^j)(\partial_j V^i) < 0, \qquad p\Big|_{\partial \mathcal{D}} = 0 \quad (14)$$

By strong maximum principle $\nabla_N p\Big|_{\partial \mathcal{D}} < 0$. Water wave problem, uniform gravitational field g



Incompressibility cond, $p > 0$ holds it together

If (13) holds then $\triangle v_i = 0$ so V is determined by its boundary values and hence one can reduce to equations on the boundary only.

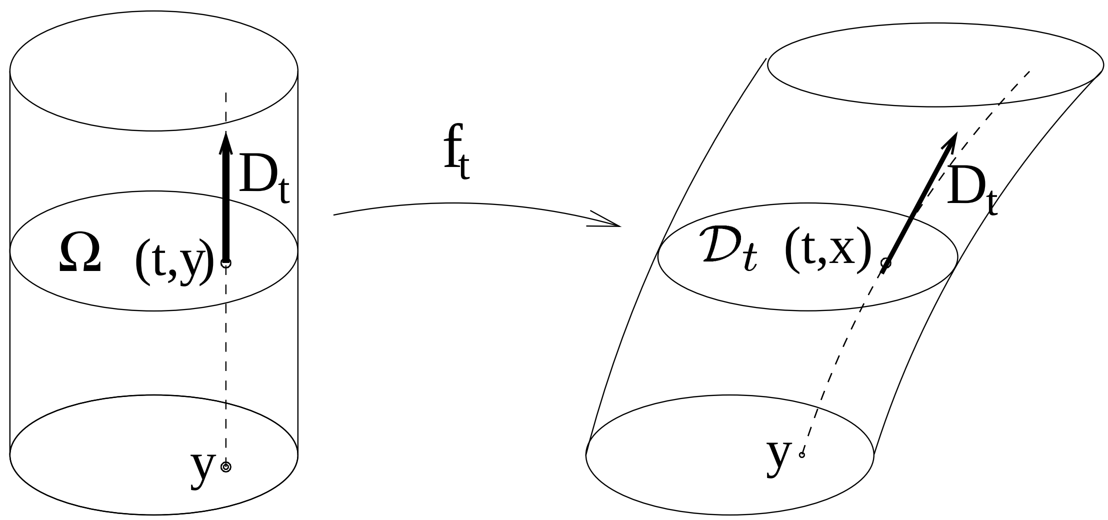
If the boundary was smooth, then inverting (14) would give that $\partial p = O(V)$ and so (12) would be an O.D.E. $(\partial_t + V^k \partial_k) V = O(V)$.

In general improved eq. for $\operatorname{div} V$ and $\operatorname{curl} v$.

Lagrangian coordinates: $f_t : y \to x(t,y)$:

$$dx/dt = V(t,x), \quad x(0,y) = f_0(y),\ y \in \Omega$$

Boundary becomes fixed in the (t,y) coord.



Lagrangian (t,y)	Eulerian (t,x)
$[0,T] \times \Omega$	$\mathcal{D} = \cup_{0 \le t \le T} \{t\} \times \mathcal{D}_t$
$D_t = \partial_t$	$D_t = \partial_t + V^k \partial_k$
$\partial_k = \frac{\partial y^a}{\partial x^k} \frac{\partial}{\partial y^a}$	$\partial_k = \frac{\partial}{\partial x^k}$

Euler's eq:

$$\rho\, D_t v_i = -\partial_i p, \qquad D_t \rho + \rho \operatorname{div} V = 0$$

Coordinates: $D_t x^i = V^i$

$$D_t \kappa - \kappa \operatorname{div} V = 0, \qquad \kappa = \det(\partial x / \partial y)$$

$(D_t \det(M) = \det(M) \operatorname{tr}(M^{-1} D_t M).)$ **so:**

$$\rho\, D_t^2 x^i = -\partial_i p, \quad \rho = k/\kappa, \quad p = p(\rho)$$

Energy Conservation $E_0(t) = E_0(0)$ where

$$E_0(t) = \int_{\mathcal{D}_t} \Big(|V|^2 + Q(\rho)\Big)\rho dx, \quad Q(\rho) = 2\int \frac{p(\rho)}{\rho^2} d\rho$$

Proof of Energy conservation: We have

$$\int_{\mathcal{D}_t} h\,\rho dx = \int_{\Omega} h\,\rho\kappa dy, \quad \kappa = \det(\partial x/\partial y), \quad D_t(\rho\kappa) = 0$$

so by the above and the divergence theorem

$$\frac{d}{dt}E_0 = \int_{\mathcal{D}_t} (D_t(|V|^2 + Q(\rho)))\rho dx$$
$$= \int_{\mathcal{D}_t} (-2V^i\partial_i p + 2p\rho^{-1}D_t\rho)\, dx$$
$$= -\int_{\partial\mathcal{D}_t} 2N_i V^i p\, dS + \int_{\mathcal{D}_t} 2(\partial_i V^i)p + 2p\rho^{-1}D_t\rho dx = 0$$

by the boundary cond. and Euler's eq.

Higher order Energies

$$E_r(t) = \|v\|_{H^r(\mathcal{D}_t)} + \|\rho\|_{H^r(\mathcal{D}_t)} + \|\theta\|_{H^{r-2}(\partial\mathcal{D}_t)}$$

where $\theta_{ij} = \bar{\partial}_i N_j$ is the second fundamental form of $\partial\mathcal{D}_t$.

Energy bound: If $\nabla_N p \leq -c_0 < 0$ then

$$E_r(t) \leq C_r(t, c_0^{-1})E_r(0).$$

Euler's eq. If $h(\rho)$ the enthalphy ($h' = p'/\rho$):

$$D_t^2 x^i + \partial_i h = 0, \quad \rho = k/\kappa, \quad h\Big|_{\partial\Omega} = 0$$

where $\kappa = \det(\partial x/\partial y)$, $x(t, y)$, $\partial_i = (\partial y^a/\partial x^i)\partial/\partial y^a$.

Linearized equations Consider a family of solutions $x = \bar{x}(t, y, r)$ depending on an extra parameter r and let $\delta x = \partial\bar{x}(t, y, r)/\partial r\Big|_{r=0}$.

$$D_t^2 \delta x^i + \partial_i \delta h - (\partial_k h)\partial_i \delta x^k = 0,$$
$$\delta h = -h'\rho\,\mathrm{div}\delta x, \quad \delta h\Big|_{\partial\Omega} = 0.$$

since $[\delta, \partial_i] = -(\partial_i \delta x^k)\partial_k$ and $\delta\kappa = \kappa\mathrm{div}\delta x$.

Linearized stability: Let

$$\tilde{E}_r(t) = \|\delta v\|_{H^r(\mathcal{D}_t)} + \|\delta\rho\|_{H^r(\mathcal{D}_t)} + \|\delta\theta\|_{H^{r-2}(\partial\mathcal{D}_t)}$$

and suppose that $\nabla_N h \leq -c_0 < 0$. Then

$$\tilde{E}_r(t) \leq C_r(x, t, c_0^{-1})\tilde{E}_r(0)$$

Existence for linearized eq.: Non standard because the higher order operator $-(\partial_k h)\partial_i \delta x^k$ is not elliptic. It is positive because $\nabla_N h < 0$.

Existence for Euler's eq.: Follows from invertibility and tame estimates for the linearized operator using the Nash-Moser technique.

Rewriting the **linearized equations** for $X = \delta x$:

$$\ddot{X} + CX = B(X, \dot{X}), \quad \text{div} X\Big|_{\partial\Omega} = 0$$

where B is a **bounded** operator,

$\dot{X} = \widehat{\mathcal{L}}_{D_t} X$ is a **modified time derivative**:

$$\widehat{\mathcal{L}}_{D_t} X^i = \mathcal{L}_{D_t} X^i + \text{div} V\, X^i = \kappa^{-1} \frac{\partial x^i}{\partial y^a} D_t \Big(\kappa \frac{\partial y^a}{\partial x^k} X^k\Big)$$

that preserves the divergence free condition: $\text{div}\widehat{\mathcal{L}}_{D_t} X = \widehat{D}_t \text{div} X$, where $\widehat{D}_t = D_t + \text{div} V$.

C is a **positive symmetric** operator on vector fields satisfying the boundary condition if the physical condition $\nabla_N h < 0$ hold.

$CX = -\nabla(h'\rho\, \text{div} X + (\partial_k h) X^k) = -\nabla(h' \text{div}(\rho X))$.

If $\langle X, Z\rangle = \int_{\mathcal{D}_t} X \cdot Z\, dx$, $\text{div} X\Big|_{\partial\Omega} = \text{div} Z\Big|_{\partial\Omega} = 0$:

$$\langle \rho X, CZ\rangle = \int_{\mathcal{D}_t} \text{div}(\rho X) \text{div}(\rho Z) h' dx$$
$$+ \int_{\partial\mathcal{D}_t} X_N Z_N (-\nabla_N h) \rho dS, \quad X_N = X \cdot N$$

(Here $\partial_k h\Big|_{\partial\mathcal{D}_t} = N_k \nabla_N h$, since $h\Big|_{\partial\mathcal{D}_t} = 0$.)

Energy $\tilde{E}_0 = \langle \dot{X}, \rho \dot{X} \rangle + \langle X, \rho C X \rangle$

Energy bound:

$$
\begin{aligned}
D_t \tilde{E}_0 = 2\langle \rho \dot{X}, \ddot{X} \rangle + \langle \rho \dot{X}, C X \rangle + \langle X, \rho C \dot{X} \rangle \\
+ \langle X, \rho [D_t, C] X \rangle + \text{L.O.} \\
= 2\langle \rho \dot{X}, \ddot{X} + C X \rangle + \langle X, \rho [D_t, C] X \rangle + \text{L.O.}
\end{aligned}
$$

Commutator estimate:

$$
|\langle X, \rho [D_t, C] X \rangle| \leq C \langle X, \rho\, C X \rangle
$$

where $C = \|\nabla_N D_t h / \nabla_N h(t, \cdot)\|_{L^\infty}$

It follows that $D_t \tilde{E}_0 \leq C \tilde{E}_0$ so $\tilde{E}_0(t) \leq C \tilde{E}_0(0)$.

Higher order energies

$\tilde{E}_r = \|\dot{X}\|_{H^r(\Omega)} + \|\text{div} X\|_{H^r(\Omega)} + \|X_N\|_{H^r(\partial\Omega)}$

Prove that $\tilde{E}_r(t) \leq C \tilde{E}_r(0)$.

Orthogonal projection onto divergence free vector fields Write $X = X_0 + X_1$, where

$$X_0 = PX = X - \nabla q, \quad \triangle q = \mathrm{div} X, \; q\Big|_{\partial\Omega} = 0$$

Decompose the linearized equations:

$$\ddot{X} + CX = B(X, \dot{X}), \quad \mathrm{div} X\Big|_{\partial\Omega} = 0$$

where $CX = -\nabla(h'\rho\,\mathrm{div} X + (\partial_k h)X^k)$, into a **wave equation** for the divergence

$$\widehat{D}_t^2 \mathrm{div} X - \triangle(p' \mathrm{div} X) = \triangle(X^k \partial_k h) + \mathrm{div}\ B,$$

with Dirichlet boundary cond. $\mathrm{div} X\Big|_{\partial\Omega} = 0$, and an **evolution eq.** for the divergence free part:

$$\ddot{X}_0 + AX_0 = -AX_1 - PB_2(X_1, \dot{X}_1) + PB(X, \dot{X})$$

(we used $[\widehat{\mathcal{L}}_{D_t}, P]X = O(X)$), for the operator

$$AX = PCX = P(-\nabla(X^k \partial_k h))$$

since the projection of the gradient of a function that vanishes on the boundary vanishes. Here A is symmetric and positive when $\nabla_N h < 0$:

$$\langle X, AZ \rangle = \int_{\partial\mathcal{D}_t} X_N Z_N (-\nabla_N h)\, dS, \quad \mathrm{div} X = \mathrm{div} Z = 0$$

but not elliptic!

Energies

$$\tilde{E}_0 = \langle \dot{X}_0, \dot{X}_0 \rangle + \langle X_0, AX_0 \rangle \\ + \langle \mathrm{div}\dot{X}_1, \mathrm{div}\dot{X}_1 \rangle + \langle p' \nabla \mathrm{div} X_1, \nabla \mathrm{div} X_1 \rangle$$

Estimates for the divergence free eq.:

$$\ddot{X} + AX = F, \qquad \mathrm{div} X = \mathrm{div} F = 0$$

(incompressible; $\det(\partial x/\partial y) = 1 = \rho$, $h = p$)

$$AX = -P\nabla(X^k \partial_k p)$$

$$E_r = \|\dot{X}\|_{H^r(\Omega)} + \|X_N\|_{H^r(\partial\Omega)}$$

$$E_r(t) \leq C_r(E_r(0) + \int_0^t \|F\|_{H^r(\Omega)} d\tau)$$

Lowest order energy estimate

$$E = \langle \dot{X}, \dot{X} \rangle + \langle X, AX \rangle$$

$$\dot{E} = 2\langle \dot{X}, \ddot{X} + AX \rangle + \langle X, [D_t, A]X \rangle + L.O. \leq CE,$$

using the **commutator estimate:**
$|\langle X, [D_t, A]X \rangle| \leq C\langle X, AX \rangle.$

Lie Derivatives $T\Big|_{\partial\Omega} \in T(\partial\Omega)$ and $\operatorname{div} T = 0$.

$$\mathcal{L}_T X^i = T^k \partial_k X^i - X^k \partial_k T^i$$

$\operatorname{div} \mathcal{L}_T X = 0$ if div $X = 0$.

Commutators

$$[\mathcal{L}_T, A] X^i = (\mathcal{L}_T \delta^{ij}) \delta_{jk} A X^k + A_{Tp} X^i$$

where for $f\Big|_{\partial\Omega} = 0$,

$$A_f X = -P\Big(\delta^{ij} \partial_j (X^k \partial_k f)\Big)$$

$$|\langle X, A_f X\rangle| \le C \langle X, AX\rangle,$$

where $C = \|\nabla_N f / \nabla_N p\|_{L^\infty(\partial\Omega)}$.

Energies $\mathcal{T}$ family of vector fields that span $T(\partial\Omega)$.

$$E_r^{\mathcal{T}}(t) = \sum_{|I|\le r,\, I\in\mathcal{T}} \sqrt{\langle \mathcal{L}_T^I \dot{X}, \mathcal{L}_T^I \dot{X}\rangle + \langle \mathcal{L}_T^I X, A \mathcal{L}_T^I X\rangle}$$

$E_r^{\mathcal{T}}(t) \le C E_r^{\mathcal{T}}(0)$.

Estimates of derivatives by the curl, the divergence and tangential derivatives:

$$|\partial Z| \leq C\Big(|\mathrm{div}\, Z| + |\mathrm{curl}\, \underline{Z}| + \sum_{S\in\mathcal{S}} |SZ|\Big)$$

$\mathcal{S}$ span $T(\partial\Omega)$

Estimates for the curl

$$\mathcal{L}_{D_t}\mathrm{curl}\, v = 0$$

$\mathcal{L}_{D_t}\mathrm{curl}\, \delta z = 0, \qquad \delta z_i = \delta_{ij}\dot{X}^j - \mathrm{curl}\, v_{ij}\, X^j$

since $\mathrm{curl}\, AX = 0$.

$$\mathcal{L}_{D_t}\mathrm{curl}\, \mathcal{L}_T^I \delta z = 0,$$

Existence for the divergence free eq.: Replace A by a a sequence of bounded operators A^ε for which existence is known and such that we uniformly have the same commutator estimates and hence energy estimates as $\varepsilon \to 0$.
Let $\chi_\varepsilon(s)=\chi(s/\varepsilon)$, where $\chi(s)=1$, when $s \geq 1$, $\chi(s) = 0$, when $s \leq 0$, and $\chi'(s) \geq 0$ and set

$$\begin{aligned} A^\varepsilon X &= -P(\chi_\varepsilon(h)\nabla(X^k \partial_k h)) \\ &= P(\chi_\varepsilon'(h)(\nabla h) X^k \partial_k h) \end{aligned}$$

The equality follows since $P\nabla\Big(\chi_\varepsilon(h) X^k \partial_k h\Big)=0$ since we project along gradients of functions that vanish on the boundary.
The equation

$$\ddot{X}^\varepsilon + A^\varepsilon X^\varepsilon = F$$

is an O.D.E. since A^ε is bounded so existence follows and one prove that with

$$E_r^\varepsilon = \|\dot{X}^\varepsilon\|_{H^r(\Omega)} + \|X_N^\varepsilon\|_{H^r(\partial\Omega)}$$

we have

$$E_r^\varepsilon(t) \leq C_r(E_r^\varepsilon(0) + \int_0^t \|F\|_{H^r(\Omega)} d\tau)$$

where C_r is independent of ε.

Inverse Function Theorems

Th. 1 Suppose that Φ is a smooth map between Banach spaces (e.g. C^k or H^k). Suppose also that $\Phi(0) = 0$ and $\Phi'(0)$ is invertible. Then for f close to 0 the equation $\Phi(x) = f$ has a solution x.

Th. 2 Suppose that Φ is a smooth tame map between tame Frechet spaces (e.g. C^∞). Suppose also that $\Phi(0) = 0$, $\Phi'(x)$ is invertible for x close to 0 and the inverse $\Phi'(x)^{-1}$ is a smooth tame map. Then for f close to 0 the equation $\Phi(x) = f$ has a solution x.

Def tame Frechet space: exist grading of seminnorms $\|g\|_a \leq \|g\|_b$, if $a \leq b$, and exist smoothing operators; S_θ, $1 < \theta < \infty$, satisfying

$$\|S_\theta u\|_b \leq \theta^{b-a}\|u\|_a, \quad \|(I - S_\theta)u\|_a \leq \theta^{a-b}\|u\|_b,$$

for $a \leq b$. P is a tame map if there is an r_0 such that for all r: $\|P(g)\|_r \leq C_r(\|g\|_{r+r_0} + 1)$.

Nash-Moser technique to solve $\Phi(x) = f$. Given x solve for δx so $\Phi(x) + \Phi'(x)\delta x = f$. Gives $\widehat{x} = x + \delta x$ so $\Phi(\widehat{x}) = f + O(\delta x)^2$. Going from x to $\widehat{x}$ looses regularity so smooth $\widehat{x}$.

The Nash-Moser technique (incompressible)

The nonlinear map: $x(t,y)\in C^\infty([0,T]\times\Omega)$

$$\Phi_i(x)=D_t^2 x_i+\partial_i p,\qquad \partial_i=(\partial y^a/\partial x^i)\partial_a,$$

where $p=\Psi(x)$ is given by solving

$$\triangle p=-(\partial_i V^k)\partial_k V^i,\quad p\Big|_{\partial\Omega}=0,\quad V=D_t x.$$

Solution of Euler's eq.

$$\Phi(x)=0,\qquad x\Big|_{t=0}=f_0,\quad D_t x\Big|_{t=0}=V_0$$

Turning initial cond. into a small inhom.
Formal power series solution x_0 as $t\to 0$, $k\geq 0$:

$$D_t^k\Phi(x_0)\Big|_{t=0}=0,\quad x_0\Big|_{t=0}=f_0,\quad D_t x_0\Big|_{t=0}=V_0$$

Let $F_0=\Phi(x_0)$, $t\geq 0$ and $F_0=0$, $t\leq 0$, $F_\delta(t,y)=F_0(t-\delta,y)$, $\tilde{\Phi}(u)=\Phi(u+x_0)-\Phi(x_0)$.

$$\tilde{\Phi}(u)=F_\delta-F_0,\qquad u\Big|_{t=0}=D_t u\Big|_{t=0}=0$$

is equiv. to $\Phi(u+x_0)=0$ for $0\leq t\leq\delta$. $\tilde{\Phi}(0)=0$ and $F_\delta-F_0\to 0$, when $\delta\to 0$.

Th Suppose that x and $\delta\Phi$ are smooth. Then

$$\Phi'(x)\delta x = \delta\Phi, \quad \delta x\Big|_{t=0} = D_t\delta x\Big|_{t=0} = 0$$

has a smooth solution δx that satisfies

$$\|D_r\delta x\|_{r-1} + \|\delta x\|_r \\ \leq K_r \int_0^t (\|\delta\Phi\|_r + |||x|||_{r+4,2}\|\delta\Phi\|_0)\, d\tau$$

if the **coordinate and physical condition** hold, where $K_r = K_r(\|x\|_{4,2})$. Here

$$\|X\|_r = \|X(t,\cdot)\|_{H^r(\Omega)}, \qquad \|X\|_{r,\infty} = \|X\|_{C^r(\overline{\Omega})}$$

$$|||X|||_{r,k} = \sup_{0\leq t\leq T} \|X\|_{r,\infty} + ... + \|D_t^k X\|_{r,\infty}$$

Include **time derivatives** up to highest or fixed order? Smoothing in space or space-time? Using Sobolev's lemma and the eq. $\Phi'(x)\delta x = D_t^2\delta x - \partial_k p\, \partial_i \delta x^k + \delta p$. we get the **tame estimate**

$$|||\delta x|||_{r,2} \leq K_r(|||\delta\Phi|||_{r+r_0,0} + |||x|||_{r+r_0+4,2}|||\delta\Phi|||_{0,0})$$

where $r_0 = [n/2] + 1$.

The coordinate and physical conditions

Let $M(t) = \sup_{y\in\Omega} \sqrt{|\partial x/\partial y|^2 + |\partial y/\partial x|^2}$. Then

$$M(t) \le 2M(0), \quad \text{for } t \le T,$$
$$\text{if} \quad T|||\dot{x}|||_1 M(0) \le 1/8$$

Let $N(t) = \sup_{y\in\partial\Omega} |\nabla_N p|^{-1}$. Then assuming that T is so small that the above hold we have

$$N(t) \le 2N(0) \quad \text{for } t \le T,$$
$$\text{if} \quad T|||\dot{p}|||_1 M(0)N(0) \le 1/8$$

Each iterate x as well as smoothing of it $S_\theta x$ will stay in the set $|||x|||_{4,2} \le 1$. Must be able to invert $\Phi'(S_\theta x)$.

Hölder norms

$$\|u\|_{a,\infty} = \sup_{x,y\in B} \sum_{|\alpha|=k} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^{a-k}} + \sup_{x\in B} |u(x)|$$

Satisfy $\|g\|_a \leq \|g\|_b$, if $a \leq b$

Smoothing operators S_θ, $1 < \theta < \infty$:

$$\|S_\theta u\|_b \leq \theta^{b-a}\|u\|_a, \quad \|(I - S_\theta)u\|_a \leq \theta^{a-b}\|u\|_b,$$

for $a \leq b$. We can take $\Omega = \{x \in \mathbf{R}^n;\ |x| \leq 1\}$. Smoothing operators exist for functions supported in the interior of a compact set, say $B_2 = \{x \in \mathbf{R}^n;\ |x| < 2\}$ Therefore we first extend our functions in $C^\infty(\overline{\Omega})$ to functions in $C_0^\infty(B_2)$. Using Stein's extension operator one can do so without changing the Hölder norms with more than a multiplicative constant.

Alternatively Smoothing in time as well. Can preserve the condition that the $x - x_0$ to infinite order as $t \to 0$, under a smoothing process. This is used in the compressible case.

Regularity properties of the Euler map

Suppose that $x \in \mathcal{F} = C^\infty\Big([0,T] \times \overline{\Omega}\Big)$ and $w_j \in \mathcal{F}$, for $j \leq k$. Set $\overline{x} = x + r_1 w_1 + ... + r_k w_k$ and suppose that $\Phi(\overline{x})$ is a C^k function of $(r_1, ..., r_k)$ close to $(0, ..., 0)$ with values in $\mathcal{F}$. We now define the k:th (directional) derivative of Φ at the point x in the directions w_i, $i = 1, .., k$ by

$$\Phi^{(k)}(x)(w_1, ..., w_k) = \frac{\partial}{\partial r_1} \cdots \frac{\partial}{\partial r_k} \Phi(\overline{x})\Big|_{r_1 = ... = r_k = 0}$$

We say that $\Phi(x)$ is differentiable at x if $\Phi(\overline{x})$ is a C^k function of $(r_1, ..., r_k)$ close to $(0, ..., 0)$ with values in $\mathcal{F}$, and if $\Phi^{(j)}(x)(w_1, ..., w_j)$ is linear in each of $w_1, ..., w_j$, for $j \leq k$. **Need:**

$$(\Phi'(u_i) - \Phi'(S_i u_i))\delta u_i = \int_0^1 \Phi''(S_i u_i + s(I - S_i)u_i)(u_i - S_i u_i, \delta u_i)\, ds$$

$$\Phi(u_{i+1}) - \Phi(u_i) - \Phi'(u_i)\delta u_i = \int_0^1 (1 - s)\Phi''(u_i + s\delta u_i)(\delta u_i, \delta u_i)\, ds$$

Tame estimate for the second derivative Φ is twice differentiable and satisfies

$$
\begin{aligned}
&|||\Phi''(u)(v_1, v_2)|||_a \\
&\quad \leq C_a\Big(|||v_1|||_{a+\mu,2}|||v_2|||_{\mu,2} + |||v_1|||_{\mu,2}|||v_2|||_{a+\mu,2}\Big) \\
&\qquad\qquad\qquad + C_a\Big(|||u|||_{a+\mu,2}|||v_1|||_{\mu,2}|||v_2|||_{\mu,2}\Big)
\end{aligned}
$$

provided that $|||x|||_{4,2} \leq 1$.