The weak null condition and global existence for Einstein's equations

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The Einstein vacuum equations determine a 4-d manifold $\mathcal{M}$ with a Lorentzian metric g, sign$(-1,1,1,1)$, with vanishing Ricci curvature:

$$R_{\mu\nu} = 0$$

The initial value problem: Given a 3-d manifold Σ, with Riemannian metric g_0, and a symmetric two-tensor k_0, we want to find a 4-d manifold $\mathcal{M}$, with a Lorentzian metric g satisfying the Einstein equations, and an imbedding $\Sigma \subset \mathcal{M}$ such that g_0 is the restriction of g to Σ and k_0 is the second fundamental form of Σ.

The initial data problem is over determined and data must satisfy **the constraint equations**:

$$R_0 - {k_0}_j^i {k_0}_i^j + {k_0}_i^i {k_0}_j^j = 0, \quad \nabla^j k_{0ij} - \nabla_i {k_0}_j^j = 0.$$

Here R_0 is the scalar curvature of g_0 and ∇ is covariant differentiation with respect to g_0.

Einstein's equations are invariant under diffeomorphisms. Eliminate this freedom by fixing a **gauge condition or system of coordinates**.

Harmonic coordinates or **wave coordinates** are given as solutions of the wave equations

$$\Box_g \, x^\mu = 0,$$

where the **geometric wave operator** is*

$$\Box_g = \nabla_\alpha \nabla^\alpha = g^{\alpha\beta}\partial_\alpha\partial_\beta + g^{\alpha\beta}\Gamma_{\alpha\beta}^{\ \ \nu}\partial_\nu$$

Here $g^{\alpha\beta}$ is the inverse of the metric, and $\Gamma_{\alpha\beta}^{\ \ \nu}$ are the Christoffel symbols for the metric. We can locally find wave coord. so $x^0 = 0$ on Σ

The metric in **wave coordinates** satisfy

$$g^{\alpha\beta}\Gamma_{\alpha\beta}^{\ \ \nu} = g^{\alpha\beta}\partial_\beta g_{\alpha\mu} - \tfrac{1}{2} g^{\alpha\beta}\partial_\mu g_{\alpha\beta} = 0. \qquad (1)$$

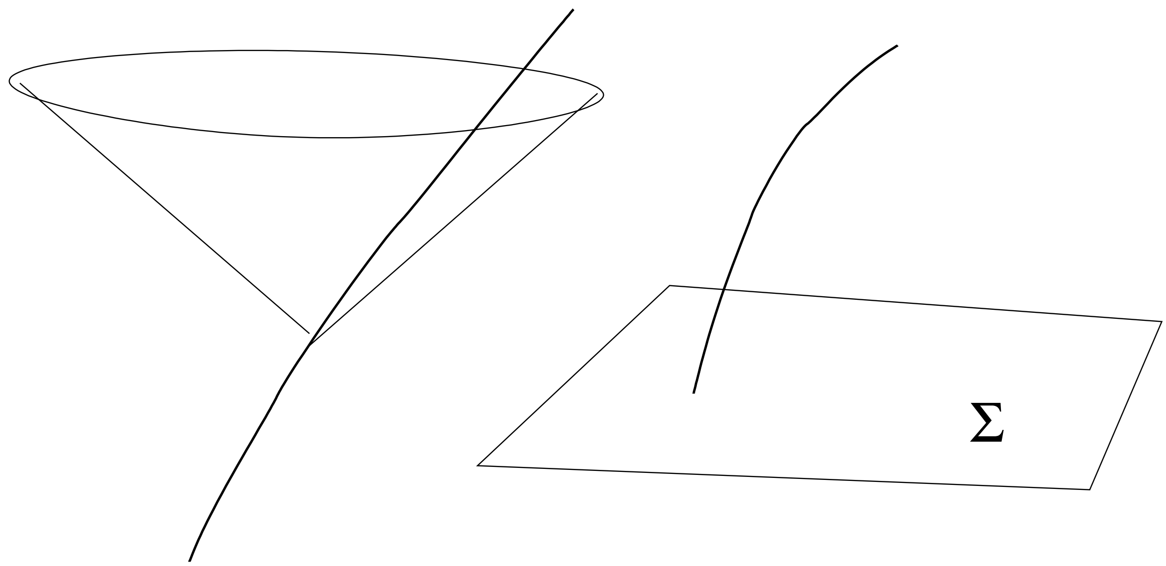
In wave coordinates, the vacuum **Einstein equations** are a system of nonlinear wave equations

$$g^{\alpha\beta}\partial_\alpha\partial_\beta \, g_{\mu\nu} = F_{\mu\nu}(g)(\partial g, \partial g), \qquad (2)$$

with $F(u)(v, v)$ depending quadratically on v. ((1) is preserved under (2).)

Local Existence for (2): Chouquet-Bruhat 1952

*We use the summation convention over repeated indices. $\partial_\alpha = \partial / \partial x^\alpha$ and Greek indices $\alpha, \beta, \mu, \nu ... = 0, ..., 3$



Causal curve $x(s)$; $g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \leq 0$, $\dot{x} = dx/ds$
Future: $\dot{x}^0 > 0$, Past: $\dot{x}^0 < 0$.

Globally hyperbolic space-times:
Every inextendable causal curve intersects the initial surface Σ once and only once.
(Any sol. constructed using an evolution eq.)

Local solution: Globally hyperbolic, smooth

Future casually geodesically complete
Future geodesics $x(s)$; $g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = const \leq 0$, $\dot{x}^0 > 0$, can be extended forever $0 \leq s < \infty$.

Global Solution: Globally hyperbolic and future causally geodesically complete, smooth.

Global stability of Minkowski space
Christodoulou-Klainerman (CK)

Constructing a global solution from initial data which is close to and asymptotically approaching the Minkowski metric $m = \text{diag}(-1, 1, 1, 1)$.

Smallness assumption on data (Σ, g_0, k_0): Σ is diffeomorphic to $\mathbf{R}^3$ and data are close to the data for Minkowski space $(\mathbf{R}^3, \delta, 0)$.

Initial data $(\mathbf{R}^3, g_0, k_0)$ are **asymptotically flat**:

$g_{0ij} = (1+2M/r)\delta_{ij} + o(r^{-1-\varepsilon}), \quad k_0 = o(r^{-2-\varepsilon}),$

$\varepsilon > 0$, when $r = |x| \to \infty$. Here $M > 0$ by the positive mass theorem. CK assumed $\varepsilon > 1/2$.

Other, restricted, global existence results: Friedrich, Klainerman-Nicolo

All proofs avoid using wave coordinates; it was believed that these would be badly behaved in the large and possibly blow-up even if in a coordinate invariant formulation the curvature remained bounded. We will come back to this.

Global Existence in the wave coordinates
L-Rodinanski (LR)

Much simpler proof, ~ 50 pages instead of 500.
Works for general small asymptotically flat data.
Also works coupled to matter fields $R_{\mu\nu} = T_{\mu\nu}$.

The metric approaches the Minkowski metric, but we presently not get as detailed information about the asymptotic behavior as CK.

CK equation for curvature, no global coord.
LR global equation for metric.

CK no explicit null condition.
LR weak null condition.
(cancelation that makes it more likely it has global exist than generic eqns of the same form)

CK constructs vector fields tangential to the curved light cones which a priori are unknown.
LR use the vector fields of the Minkowski cones.

Einstein's equations in wave coordinates

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\, g_{\mu\nu} = F_{\mu\nu}(g)(\partial g, \partial g), \qquad (3)$$

Stability around the Minkowski metric $m = \text{diag}(-1,1,1,1)$; $h_{\mu\nu} = g_{\mu\nu} - m_{\mu\nu}$ is small.
Generic **systems of wave equations:**

$\Box\phi_I = c^{JK}_{I\alpha\beta}\,\phi_J\,\partial_\alpha\partial_\beta\phi_K + d^{JK}_{I\alpha\beta}\,\partial_\alpha\phi_J\,\partial_\beta\phi_K +$ cubic with small initial data.(Here $\Box = m^{\alpha\beta}\partial_\alpha\partial_\beta$.)

Blow up for small data (John) e.g. $\Box\phi = (\partial_t\phi)^2$.
Global existence if $c^{JK}_{I\alpha\beta} = 0$ and $d^{JK}_{I\alpha\beta}$ satisfies the **null condition** (Christodoulou,Klainerman) e.g.$\Box\phi = (\partial_t\phi)^2 - |\nabla_x\phi|^2$. Need εt^{-1} decay to handle general quadratic nonlinear terms.
Problem:(3) does not satisfy the null condition

(3) satisfy the **weak null condition**(LR).
Essentially the system decouples in a **null-frame**

$$\Box\phi_2 = (\partial_t\phi_1)^2, \qquad \Box\phi_1 = 0$$

$\partial\phi_1 \sim \varepsilon t^{-1}$, $\partial\phi_2 \sim \varepsilon t^{-1}\ln|t|$

Global existence (L-radial case, Alinhac-general) $\Box\phi = \phi\triangle\phi$ but solutions only decay like $\varepsilon t^{-1+c\varepsilon}$.

The weak null condition detect situations where the asymptotic behavior is not free.

The weak null condition for a generic system

$$\Box\phi_I = \sum A^{JK}_{I,\alpha\beta}\,\partial^\alpha\phi_J\,\partial^\beta\phi_K + \text{cubic terms} \quad (4)$$

is that the asymptotic system for $\Phi_I = r\phi_I$:

$$(\partial_t + \partial_r)(\partial_t - \partial_r)\Phi_I \sim r^{-1}\sum A^{JK}_{I,nm}(\partial_t - \partial_r)^n\Phi_J\,(\partial_t - \partial_r)^m\Phi_K, \quad (5)$$

has global solutions for all small data. Here,

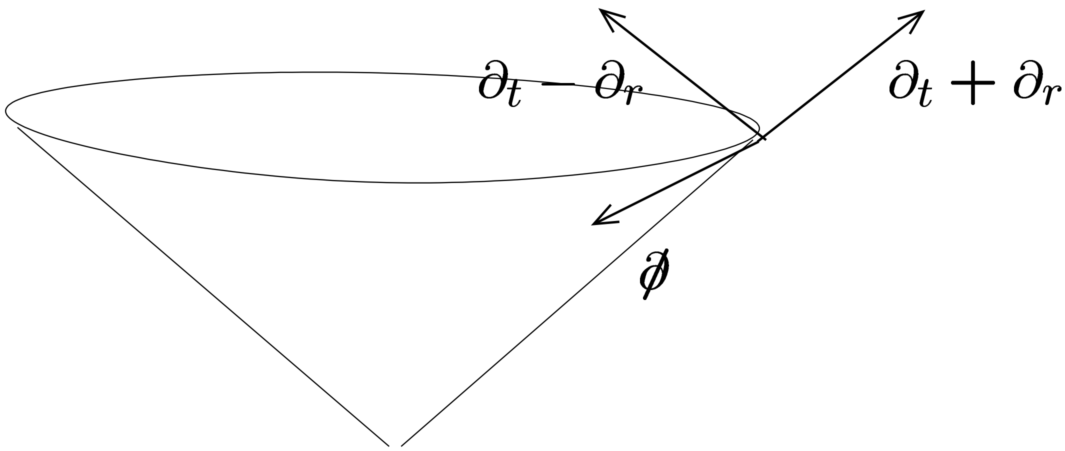
$$A^{JK}_{I,nm} = \sum_{|\alpha|=n,|\beta|=m} \frac{1}{(-2)^{m+n}} A^{JK}_{I,\alpha\beta}\,\widehat{\omega}^\alpha\widehat{\omega}^\beta, \quad \widehat{\omega} = (-1,\omega),\ \omega \in \mathbf{S}^2$$

The usual null condition is that $A^{JK}_{I,nm}(\omega) \equiv 0$. The asymptotic system was introduced by Hörmander to find the time of blow-up.

The asymptotic system (5) is obtained from the system (4) by neglecting derivatives tangential to the outgoing Minkowski light cones; $t = |x|$, and cubic terms, that are decaying faster

$$\Box\phi = r^{-1}(\partial_t + \partial_r)(\partial_t - \partial_r)(r\phi) + \text{angular derivatives}$$

$$\partial_\mu = -\tfrac{1}{2}\widehat{\omega}_\mu(\partial_t - \partial_r) + \text{tangential derivatives } \overline{\partial}_\mu$$



$$\Box\phi = 0 \implies |\partial\phi| \leq C/t \text{ and } |\bar\partial\phi| + |(\partial_t+\partial_r)\phi| \leq C/t^2$$

A simple example of a system satisfying the weak null condition, violating the standard null condition and yet possessing global solutions is

$$\begin{aligned} \Box\phi_1 &= \phi_3 \cdot \partial^2\phi_1 + (\partial\phi_2)^2, \\ \Box\phi_2 &= 0, \qquad \Box\phi_3 = 0. \end{aligned} \tag{6}$$

Another example is provided by the equation

$$\Box\phi = \phi\Delta\phi. \tag{7}$$

The proof of small data global existence for this is very involved, [L-](radial case),[Alinhac]

The asymptotic system for Einstein's eq. can be modelled by that of (6).

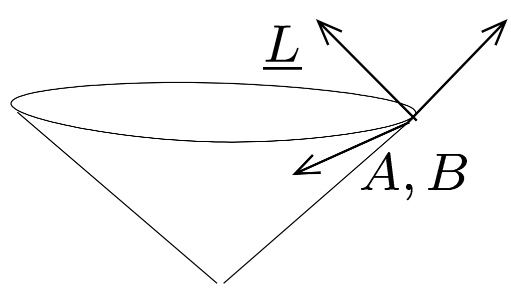
We will refer to $\phi \cdot \partial^2\phi$ as the **quasilinear** terms and $\partial\phi \cdot \partial\phi$ as the **semilinear** terms.

Einstein's eq. $h = g - m$ small

$$\widetilde{\Box}_g h_{\mu\nu} = g^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h) =$$
$$= P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h),$$

where $Q_{\mu\nu}$ are linear combinations of the standard null-forms and $G_{\mu\nu}(h)(\partial h, \partial h)$ is cubic,

$$P(k,p) = \frac{1}{4}\,\mathrm{tr}\,k\ \mathrm{tr}\,p - \frac{1}{2}k^{\alpha\beta}p_{\alpha\beta}, \quad \mathrm{tr}\,k = m^{\alpha\beta}k_{\alpha\beta}$$



Null-frame decomposition of Einstein's eq. We define a null-frame of vectors by $\underline{L} = (-1, \omega)$, $L = (1, \omega)$ and $A, B \in \mathbf{S}^2$ such that $\mathcal{T} = \{L, A, B\}$ span the tangent of the outgoing light cones and $\mathcal{U} = \{A, B, L, \underline{L}\}$ span the full tangent space.

In terms of the null-frame we have

$$P(\ell, k) = c^{ijkl}\,\ell_{T_i U_j} \cdot k_{T_k U_l} - \frac{1}{8}(\ell_{LL}k_{\underline{LL}} + \ell_{\underline{LL}}k_{LL})$$

where the sum is over $T_i \in \mathcal{T}$ and $U_i \in \mathcal{U}$.

Parity cond. For each $\underline{L}$ comp. there is an L.

The asymptotic system for Einstein's eq.

$$\widetilde{\square}_g h_{\mu\nu} \sim \tfrac{1}{4}\widehat{\omega}_\mu\widehat{\omega}_\nu P(\underline{\partial} h, \underline{\partial} h), \quad \underline{\partial} = \partial_t - \partial_r$$
$$P(\partial h, \partial h) = \underline{\partial} h_{TU} \cdot \underline{\partial} h_{TU} - \tfrac{1}{4}\underline{\partial} h_{LL}\, \underline{\partial} h_{\underline{LL}}$$

Then $T^\mu\widehat{\omega}_\mu = 0$, for $T \in \mathcal{T}$ and $\underline{L}^\mu\widehat{\omega}_\mu = 2$ so

$$(\widetilde{\square}_g h)_{\underline{L}\,\underline{L}} \sim P(\underline{\partial} h, \underline{\partial} h)$$
$$(\widetilde{\square}_g h)_{TU} \sim 0, \qquad T \in \mathcal{T},\ U \in \mathcal{U}$$

Asymptotic form of **wave coordinate cond.**

$$\underline{\partial} h_{LT} \sim 0, \qquad T \in \mathcal{T} = \{L, A, B\} \qquad (8)$$

Hence as far as the **semilinear** terms it looks like the system $\square\phi_2 = (\partial\phi_1)^2$ where $\square\phi_1 = 0$.

The **quasilinear** part: Since $g_{\alpha\beta} = m_{\alpha\beta} + h_{\alpha\beta}$ it follows that the inverse $g^{\alpha\beta} = m^{\alpha\beta} - h^{\alpha\beta} + O(h^2)$;

$$\widetilde{\square}_g = g^{\alpha\beta}\partial_\alpha\partial_\beta \sim \square - \tfrac{1}{4}h_{LL}\underline{\partial}^2,$$

so $(\square h)_{LL} \sim \frac{1}{4}h_{LL}\underline{\partial}^2 h_{LL}$. It appears as bad as $\square\phi = \phi\triangle\phi$, but because of (8) $h_{LL} \sim M/r$.

For Einstein's eq. the whole energy tensor has to be estimated together but for the asymptotic system we can separate the components.

What is used in the proof:

The **Klainerman-Sobolev inequality** ($r = |x|$)

$$|\phi(t,x)| \le \frac{C\sum_{|I|\le 2}\|Z^I\phi(\tau,\cdot)\|_{L^2}}{(1+t+r)(1+|t-r|)^{1/2}},$$

where Z^I is a product of $|I|$ vector fields of the form ∂_i, $x_i\partial_j - x_j\partial_i$, $t\partial_i + x_i\partial_t$ that commute with $\Box$ and $t\partial_t + x^i\partial_i$; $[\Box, Z] = c_Z\Box$.

Decay estimate

$$\|(1+t)\partial\phi(t,\cdot)\|_{L^\infty} \le C\int_0^t (1+\tau)\|\tilde{\Box}_g\phi(\tau,\cdot)\|_{L^\infty} d\tau + C\int_0^t \sum_{|I|\le 2}\|Z^I\phi(\tau,\cdot)\|_{L^\infty}\frac{d\tau}{1+\tau}$$

The estimate above is in philosophy somewhat related to the asymptotic eq. in the sense that it is obtained by regarding the angular derivatives as lower order and integrating the eq.

$$\tfrac{1}{r}(\partial_t+\partial_r)(\partial_t-\partial_r)(r\phi) = \Box\phi + \tfrac{1}{r^2}\triangle_\omega\phi$$

Generalized energy inequality

$$\int_{\Sigma_T} |\partial\phi|^2 + \int_0^T \int_{\Sigma_\tau} \frac{\gamma\ |\overline{\partial}\phi|^2}{(1+|t-r|)^{1+2\gamma}}$$
$$\le 8\int_{\Sigma_0}|\partial\phi|^2 + C\varepsilon\int_0^T\int_{\Sigma_t}\frac{|\partial\phi|^2}{1+t} + 16\int_0^T\int_{\Sigma_t}|\widetilde{\Box}_g\phi||\partial_t\phi|$$

The commutator estimates If $[Z,\Box]=0$ then

$$|[Z,\widetilde{\Box}_g]| = \Big|[Z, g^{\alpha\beta}\partial_\alpha\partial_\beta]\phi\Big| \le \frac{C\varepsilon}{1+t+r}\sum_{|I|\le 1}|\partial Z^I\phi|,$$

The decay, the energy and the commutator estimates for $\widetilde{\Box}_g$ hold under some weak decay obtained from the K-S ineq. and strong decay:

$$|h|_{L\mathcal{T}} + |Zh|_{LL} \le C\varepsilon(1+|t-r|)/(1+t+r)$$

obtained from the **wave coordinate cond.** C-K for Einstein's eq. and Alinhac for $\Box\phi = \phi\triangle\phi$ had to modify the vector fields at infinity in order to get good commutators. For us the wave coordinate gauge makes the geometry of the characteristic surfaces be close that of the Minkowski, or rather the Schwarzschild, ones.

Structure of the proof:

Energies $E(t) = \sum_{|I|\leq N} \|\partial Z^I \phi(t,\cdot)\|_{L^2}$.

Assuming decay like $|\partial\phi| \leq C\varepsilon(1+t)^{-1}$ for some components the Gen. energy ineq.

$$E(t) \leq E(0) + \int_0^t C\varepsilon(1+\tau)^{-1} E(\tau)\, d\tau + ...$$

gives **energy estimate** $E(t) \leq (1+t)^{C\varepsilon}$.

The energy est. and the K-S ineq. gives **weak decay est.** $|\partial\phi| \leq C\varepsilon(1+t)^{-1+C\varepsilon}(1+|t-r|)^{-1/2}$

Integrating the weak decay est. applied to $Z\phi$ gives full decay for derivatives tangential to the outgoing Minkowski light cones
$|\overline{\partial}\phi| \leq C\varepsilon(1+t)^{-2+C\varepsilon}(1+|t-r|)^{1/2}$

Nonlinear Estimates using the equations and the weak decay estimates gives **strong decay estimates** $|\partial\phi| \leq C\varepsilon(1+t)^{-1}$ for some comp.

The strong decay estimates can then be feed back into the energy inequality.

The semilinear model $\Box\phi_1 = (\partial\phi_2)^2$, $\Box\phi_2 = 0$

Bootstrap assumption: $E(t) \leq E(0)(1+t)^{1/4}$

We will use this to derive $E(t) \leq E(0)(1+t)^{C\varepsilon}$

If $\Box\phi = 0$ with compact support data then $\phi = 0$ when $r \geq t + C$, and integrating the K-S ineq.

$|Z^J\phi(t,x)| \leq CE(t)(1+t)^{-1/2} \leq CE(0)(1+t)^{-1/4}$

and by the **decay estimate**

$$\|(1+t)\partial\phi_i(t,\cdot)\|_{L^\infty} \leq C\int_0^t (1+\tau)\|\Box\phi_i(\tau,\cdot)\|_{L^\infty} d\tau + C\int_0^t \sum_{|I|\leq 2} \|Z^I\phi_i(\tau,\cdot)\|_{L^\infty} \frac{d\tau}{1+\tau}$$

we get

$$\|\partial\phi_2(t,\cdot)\|_{L^\infty} \leq \frac{CE(0)}{1+t}$$

$$\|\partial\phi_1(t,\cdot)\|_{L^\infty} \leq \frac{CE(0)}{1+t} + (1+t)^{-1}\int_0^t \|\partial\phi_2(\tau,\cdot)\|^2_{L^\infty}(1+\tau)\, d\tau \leq \frac{CE(0)}{1+t} + \frac{CE(0)^2 \ln(1+t)}{1+t}$$

Wave coordinate condition $(h = g - m)$

$$\partial_\mu h^\mu{}_\nu = -\tfrac{1}{2}\partial_\nu \,\mathrm{tr}\, h + h\cdot\partial h$$

Expand in a null-frame:

$$\partial_{\underline{L}} h^{\underline{L}}{}_\nu + \partial_L h^L{}_\nu + \partial_A h^A{}_\nu = -\tfrac{1}{2}\partial_\nu \,\mathrm{tr}\, h + h\cdot\partial h$$

Hence $\partial_{\underline{L}} h_{LT} = \overline{\partial} h + h\cdot\partial h$ so

$$|\partial h|_{LT} \le C|\overline{\partial} h| + C|h|\,|\partial h|$$

It follows that

$$|\partial h|_{LT} \le C\varepsilon t^{-2+C\varepsilon}|t-r|^{1/2}$$

and integrating this gives

$$|h|_{LT} \le C\varepsilon t^{-1}(1+|t-r|)$$

Furthermore

$$|Zh|_{LL} \le C\varepsilon t^{-1}(1+|t-r|)$$

This follows from commuting the wave coordinate condition with Z using that

$$\partial_\alpha Z_\beta = c_{\alpha\beta}\partial^\beta, \qquad \text{where} \qquad c_{LL} = 0$$

since Z is conformally killing.

Expand in a nullframe

$$|h^{\alpha\beta}\partial_\alpha\partial_\beta\phi| \le C|h|_{LL}|\partial^2\phi| + C|h||\overline{\partial}\partial\phi|$$
$$\le C\Big(\frac{|h|_{LL}}{|1+|t-r|} + \frac{|h|}{1+t}\Big)\sum_{|I|\le 1}|\partial Z^I\phi|$$

where we used the inequalities

$$|\partial f|(1+|t-r|) + |\overline{\partial} f|(1+t+r) \le C\sum_{|I|\le 1}|Z^I f|$$

Commutator estimate

$$\Big|[Z,\widetilde{\Box}_g]\phi\Big|$$
$$\le C\Big(\frac{|H|+|ZH|}{1+t+r} + \frac{|ZH|_{LL}+|H|_{L\mathcal{T}}}{1+|t-r|}\Big)\sum_{|I|\le 1}|\partial Z^I\phi|$$
$$\le \frac{C\varepsilon}{1+t+r}\sum_{|I|\le 1}|\partial Z^I\phi|$$

where we used the weak decay of all comp. and the strong decay from the wave coord. cond.