KÄHLER MANIFOLDS AND CROSS QUADRATIC BISECTIONAL CURVATURE

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Abstract. In this article we continue the study of the two curvature notions for Kähler manifolds introduced by the first named author earlier: the so-called cross quadratic bisectional curvature (CQB) and its dual ($^d$CQB). We first show that compact Kähler manifolds with $CQB_1 > 0$ or $^dCQB_1 > 0$ are Fano, while nonnegative $CQB_1$ or $^dCQB_1$ leads to a Fano manifold as well, provided that the universal cover does not contain a flat de Rham factor. For the latter statement we employ the Kähler-Ricci flow to deform the metric. We conjecture that all Kähler C-spaces will have nonnegative CQB and positive $^d$CQB. By giving irreducible such examples with arbitrarily large second Betti numbers we show that the positivity of these two curvature put no restriction on the Betti number. A strengthened conjecture is that any Kähler C-space will actually have positive CQB unless it is a $P^1$ bundle. Finally we give an example of non-symmetric, irreducible Kähler C-space with $b_2 > 1$ and positive CQB, as well as compact non-locally symmetric Kähler manifolds with CQB $< 0$ and $^d$CQB $< 0$.

1. Introduction

In a recent work [18] by the first named author, the concept of cross quadratic bisectional curvature (denoted as CQB from now on) and its dual notion (denoted by $^d$CQB) for Kähler manifolds were introduced (they shall be defined shortly below). Both concepts are closely related to the notion of quadratic bisectional curvature (abbreviated as QB, see [22], [4], [23], [12], [5], [19], and [18] for the definition and results related it). One of the reasons for the consideration of these different notions of curvature is to find suitable differential geometric characterizations for the Kähler C-spaces, as motivated by the generalized Hartshorne conjecture.

In [18], motivated by the connection between the positive orthogonal Ricci (denoted by $Ric^+ > 0$, and studied in [21, 20]) and QB $> 0$, among other things the first named author proved that CQB $> 0$ implies $Ric^+ > 0$, which leads to the vanishing of holomorphic forms and simply-connectedness of the compact Kähler manifolds. The positivity of $^d$CQB, on the other hand, leads to the vanishing of the first cohomology group of the holomorphic tangent bundle, thus the manifold must be infinitesimally rigid, i.e., without nontrivial small deformations. It is also proved in [18] that, any classical Kähler C-space $M^n$ with $b_2 = 1$ and $n \geq 2$ will have positive CQB and positive $^d$CQB. This makes the two conditions (namely CQB$> 0$ and $^d$CQB$> 0$) better candidates than QB in terms of describing Kähler C-spaces, as only about eighty percent of the above spaces have positive or nonnegative QB by the excellent work of Chau and Tam [5].

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Inspired by the perspective of a curvature characterization of the Kähler C-spaces, in this paper we continue the project of understanding (with the aim of classifying) compact Kähler manifolds with positive or nonnegative CQB (or $^d$CQB). Recall that by [18], on a Kähler manifold $(M^n, g)$, if we denote by $T'M$ and $T''M$ the holomorphic and anti-holomorphic tangent bundle of $M$, then CQB is a Hermitian quadratic form on linear maps $A : T''M \to T'M$:

$$CQB_R(A) = \sum_{\alpha, \beta=1}^n R(A(E_\alpha), \overline{A(E_\alpha)}, E_\beta, E_\beta) - R(E_\alpha, E_\beta, A(E_\alpha), \overline{A(E_\beta)})$$

where $R$ is the curvature tensor of $M$ and $\{E_\alpha\}$ is a unitary frame of $T'M$. The expression is independent of the choice of the unitary frame. When the meaning is clear we simply write CQB. The manifold $(M^n, g)$ is said to have positive (nonnegative) CQB, if at any point $x \in M$, and for any non-trivial linear map $A : T'_xM \to T''_xM$, the value $CQB(A)$ is positive (nonnegative). We say that $CQB_k > 0$ if (1.1) holds for all $A$ with rank no greater than $k$.

Similarly, the dual notion ($^d$CQB) introduced in [18] is a Hermitian quadratic form on linear maps $A : T'M \to T''M$:

$$^dCQB_R(A) = \sum_{\alpha, \beta=1}^n R(A(E_\alpha), A(E_\alpha), E_\beta, E_\beta) + R(E_\alpha, E_\beta, A(E_\alpha), A(E_\beta))$$

where $R$ again is the curvature tensor of $M$ and $\{E_\alpha\}$ is a unitary frame of $T'M$. The manifold $(M^n, g)$ is said to have positive (nonnegative) $^d$CQB, if $^dCQB(A) > 0$ (or $\geq 0$) at any point in $x \in M$, and for any non-trivial linear map $A : T'_xM \to T''_xM$. Related to this there is a tensor analogous to the Ricci: $Ric^+(X, \overline{X}) = Ric(X, \overline{X}) + H(X)/|X|^2$, where $H$ is the holomorphic sectional curvature. We say that $^dCQB_k > 0$ if (1.2) holds for all $A$ with rank no greater than $k$.

It is proved in [18] that compact Kähler manifold $M^n$ with $Ric^+ > 0$ is projective and simply connected. Also, if $^dCQB > 0$, then $H^1(M, T'M) = \{0\}$, so $M$ is locally deformation rigid. Moreover $^dCQB_1 > 0$ implies $Ric^+ > 0$.

Serving as a further step of the study our first result of this article is that the positivity of either CQB$_1$ or $^d$CQB$_1$ implies the positivity of the Ricci curvature, so a compact manifold with either CQB$_1 > 0$ or $^d$CQB$_1 > 0$ is Fano, answering positively a question asked in [18].

**Theorem 1.1.** Let $(M, g)$ be a Kähler manifold with either CQB$_1 > 0$ or $^d$CQB$_1 > 0$. Then its Ricci curvature is positive. So compact Kähler manifolds with positive CQB$_1$ or $^d$CQB$_1$ are Fano.

As a corollary, the above theorem implies that a product Kähler manifold will have positive (or nonnegative) CQB or $^d$CQB if and only if each of its factors is so:

**Corollary 1.2.** Let $M = M_1 \times M_2$ be a product Kähler manifold. Then $M$ has CQB $> 0$ (or $\geq 0$) if and only if both $M_1$ and $M_2$ are so. Also, for any positive integer $k$, $M$ has CQB$_k > 0$ (or $\geq 0$) if and only if both $M_1$ and $M_2$ are so. The same statements hold for $^dCQB$ or $^dCQB_k$ as well.

By deforming the metric via the Kähler-Ricci flow we further show that if $M$ has CQB$_1 > 0$ (or $^d$CQB$_1 > 0$) and its universal cover does not contain a flat de Rham factor then $M$ is Fano as well. Note that the finiteness of the fundamental group of $M$ implies the nonexistence
of the flat de Rham factor. Namely in particular, if $M$ has $\text{CQB}_1 \geq 0$ (or $\text{dCQB}_1 \geq 0$) and $\pi_1(M)$ is finite, then $M$ is a Fano manifold:

**Theorem 1.3.** Let $(M, g)$ be a compact Kähler manifold with $\text{CQB}_1 \geq 0$ (or $\text{dCQB}_1 \geq 0$) and its universal cover does not contain a flat de Rham factor. Then $M$ is Fano. In fact, the Kähler-Ricci flow will evolve the metric $g$ to ones with positive Ricci curvature.

To prove this we adopt a nice technique of Böhm-Wilking [1] of deforming the metric via the Kähler-Ricci flow into the metric $g(t)$ with positive Ricci curvature to our curvature conditions. In [1], the authors deformed a Riemannian metric with nonnegative sectional curvature (also assuming finiteness of the fundamental group) into one with positive Ricci via the Ricci flow. Since $\text{CQB}_1 \geq 0$ (or $\text{dCQB}_1 \geq 0$) is different from the sectional curvature being nonnegative, a different collection of invariant time-dependent convex sets is constructed to serve the purpose. We also need somewhat different estimates to show that $\text{Ric}(g(t)) > 0$ for $t > 0$, where $g(t)$ is a short time solution of the Kähler-Ricci flow. In fact our curvature conditions here are much weaker than the bisectional curvature being nonnegative, which is weaker than the sectional curvature), in view of the result of Mok [15] asserting that the nonnegativity of bisectional curvature implies that the irreducible Kähler manifold is locally Hermitian symmetric, and that the first author proved in [18] that all classical Kähler C-spaces with $b_2 = 1$ admits Einstein metrics with $\text{CQB} > 0$ and $\text{dCQB} > 0$ (see also further examples with $b_2 > 1$ in this paper).

By the structure theorem for compact Kähler manifolds with nonnegative Ricci [3], we have the following:

**Corollary 1.4.** Let $(M, g)$ be a compact Kähler manifold with $\text{CQB}_1 \geq 0$ (or $\text{dCQB}_1 \geq 0$). Then there exists a finite cover of $M'$ of $M$, such that $M'$ is a holomorphic and metric fiber bundle over its Albanese variety, which is a flat complex torus, with the fiber being a Fano manifold.

Note that for a compact Kähler manifold with nonnegative QB, any harmonic $(1,1)$ form is parallel, and the positivity of QB implies that $b_2 = 1$. The positivity/nonnegativity of QB or $\text{dCQB}$ seems not to put any restrictions on $b_2$ (see Theorem 1.6 below). However, since $\text{CQB} > 0$ implies positive $\text{Ric}^\perp$ by [18], while $\mathbb{P}^1$ bundles do not admit any Kähler metric with positive $\text{Ric}^\perp$ by [20], so for Kähler C-spaces with $b_2 > 1$, we could only hope for nonnegative CQB instead of positive CQB in general. We propose the following:

**Conjecture 1.5.** Any Kähler C-space will have nonnegative CQB and positive $\text{dCQB}$.

A slightly weaker statement would be: any Kähler-Einstein C-space will have nonnegative CQB and positive $\text{dCQB}$. As a supporting evidence to Conjecture 1.5, we prove the following:

**Theorem 1.6.** There are irreducible Kähler C-spaces with arbitrarily large $b_2$ which have nonnegative CQB and positive $\text{dCQB}$.

To prove this result as an initial study towards the conjecture, we look into the simplest kind of irreducible Kähler C-spaces with $b_2 > 1$, namely, Type A flag manifolds: $M^n = SU(r+1)/T$, where $T$ is a maximal torus in $SU(r+1)$. The complex dimension is $n = \frac{1}{2}r(r+1)$ and $b_2 = r$. Equip $M^n$ with the Kähler-Einstein metric $g$, we show that it has nonnegative CQB and positive $\text{dCQB}$. This answers negatively another question asked in [18] regarding $b_2$. 
As shown in [20], any $\mathbb{P}^1$ bundle cannot admit a Kähler metric with positive orthogonal Ricci curvature, thus cannot have positive CQB. We speculate that any Kähler C-space which is not a $\mathbb{P}^1$ bundle has a metric with positive CQB.

For compact Hermitian symmetric spaces, this speculation holds true (see Corollary 2.3 in the next section). For non-symmetric Kähler C-spaces, result below gives at least one example of irreducible Kähler C-space of $b_2 > 1$ with positive CQB. Such a space is necessarily not a $\mathbb{P}^1$ bundle.

Consider irreducible Kähler C-spaces of Type A in general, namely, $SU(r+1)/K$, where $K$ is the centralizer of some sub-torus of $T$. The smallest dimensional such space which is not a $\mathbb{P}^1$ bundle and not symmetric is $M^{12} = SU(6)/S(U(2) \times U(2) \times U(2))$. It has $b_2 = 2$. Equip it with the Kähler-Einstein metric, we show that it indeed has positive CQB.

**Theorem 1.7.** Let $M^{12} = SU(6)/S(U(2) \times U(2) \times U(2))$ be the irreducible Kähler C-space which is non-symmetric, with $b_2 = 2$, and equip it with the Kähler-Einstein metric. Then it has positive CQB and positive $dCQB$.

We should point out that understanding the curvature behavior of Kähler C-spaces is a nontrivial matter, despite the fact that such spaces are classical objects of study since 1950s and are fully classified from the Lie algebraic point of view. As an illustrating example, recall the following general belief:

**Conjecture 1.8.** Any Kähler C-space has positive holomorphic sectional curvature $H$.

This question is still widely open. For Kähler C-spaces with $b_2 = 1$, all the classical types plus a few exceptional ones are known to have $H > 0$ by the work of Itoh [11]. In a recent PhD thesis [14], Simon Lohove underwent a highly sophisticated approach and he was able to show that all irreducible Kähler C-spaces of classical type with rank less than or equal to 4 have $H > 0$. Note that the rank here means that of the group, so all such spaces have $b_2 \leq 4$ in particular. Through isometric embedding, he also reduced the question largely to the case of flag manifolds with Kähler-Einstein metrics.

In the more challenging opposite direction, we propose the following:

**Conjecture 1.9.** Let $(M, g)$ be a Kähler-Einstein manifold with $CQB \geq 0$ and $dCQB > 0$. Then $M$ is biholomorphic to a Kähler C-space.

This conjecture, if affirmed, would be the first curvature characterization of compact homogeneous Kähler manifolds, which has been long missing but hoped for, in the direction of generalized Hartshorne conjecture. Of course an even bolder speculation would be to drop the Kähler-Einstein assumption in the above conjecture. The simply-connectedness, projectivity, and deformation rigidity result proved recently in [18], and Theorem 1.3 above are positive evidences towards this conjecture. Theorem 1.3 and Corollaries 1.2 and 1.4 also serve an initial step towards the classification conjecture as the main result of [10] towards the classification of Kähler manifolds with nonnegative bisectional curvature. The examples in Theorems 1.6 and 1.7 indicate that the situation here is more delicate.

Note that most results mentioned above, except the construction of examples, hold for the non-positive cases by flipping the sign of the curvature. These results are summarized in the last section. In the last section we also show that the two dimensional Mostow-Siu example [16] had $CQB < 0$ and $dCQB < 0$. This is a non-Hermitian symmetric example to which Theorem 4.1 of [18] can be applied, hence locally deformation rigid (it is in fact strongly
rigid in the sense of Sin as well). This naturally leads to the question of the role played by CQB and $d^4$CQB in the strong rigidity and holomorphicity of harmonic maps. We leave this to a future study.

2. CROSS QUADRATIC BISECTIONAL CURVATURE AND ITS DUAL

It is proved in [18] that positive CQB$_1$ implies that the orthogonal Ricci curvature $Ric^{\perp}$ is positive, and CQB$_2 > 0$ implies that the Ricci curvature $Ric$ is 2-positive, namely, the sum of any two of its eigenvalues is positive. We first show that the Ricci curvature is also positive under the CQB$_1 > 0$ assumption:

**Theorem 2.1.** Let $(M^n, g)$ $(n \geq 2)$ be a Kähler manifold with positive (or nonnegative) CQB$_1$, then its Ricci curvature is also positive (or nonnegative). Moreover $Ric(X, \overline{X}) \geq \frac{1}{n-1}Ric^{\perp}(X, \overline{X})$.

**Proof.** First we claim that, under the assumption that CQB$_1$ is positive, then for any unit vectors $X$, $Y$ in $T'M$ such that $X \perp Y$, we must have $Ric(X, \overline{X}) > R(X, \overline{X}, Y, \overline{Y})$. To see this, let $E$ be a unitary frame for $T'M$ with $X = E_1$ and $Y = E_2$, and let $A$ be the map such that $A(E_2) = E_1$ and $A(E_1) = 0$ for any $i \neq 2$. Applying (2, 1) we get $Ric_{1T} > R_{221T}$, so the claim is proved. By the same token, $Ric_{1T} > R_{211T}$ for any $i > 1$. Add up these inequalities for $i$ from 2 to $n$, we get $(n-1)Ric_{1T} > R_{111T}$, so the Ricci curvature is positive since the orthogonal Ricci is known to be positive by [18]. The nonnegative case goes similarly. \Box

**Corollary 2.2.** Let $M^n = M_1 \times M_2$ be a product Kähler manifold. Then $M$ has positive (or nonnegative) CQB$_k$ if and only if both $M_1$ and $M_2$ have positive (or nonnegative) CQB$_k$ for any $1 \leq k \leq n$.

**Proof.** Since CQB is independent of the choice of the unitary frames $E$ we take the unitary frame $E$ to be compatible with the product structure:

$$E = \{E_1, \ldots, E_r; E_{r+1}, \ldots, E_n\},$$

where $r$ is the dimension of $M_1$ and the first $r$ elements give a frame for $M_1$. We will use the index convention that $i, j, \ldots$ run from 1 and $r$, while $\alpha, \beta, \ldots$ run from $r+1$ and $n$. Denote by $R'$, $R''$ the curvature tensor of $M_1$, $M_2$, respectively, and write

$$A(E_i) = A'(E_i) + B(E_i), \quad A(E_\alpha) = C(E_\alpha) + A''(E_\alpha)$$

for the decomposition into $T'M = T'M_1 \times T'M_2$, then by definition, we have

$$CQB^M(A) = \sum_{a,b,c,d=1}^n Ric_{a\overline{\alpha}}A_{ca}\overline{A}_{\overline{\alpha}b} - \sum_{a,b,c,d=1}^n R_{a\overline{\alpha}b\overline{\alpha}}A_{ac}\overline{A}_{\overline{\alpha}d}$$

$$= \sum_{i,j,c} Ric_{i\overline{\alpha}}A_{ci}\overline{A}_{\overline{\alpha}j} + \sum_{a,\beta,\epsilon} Ric_{a\overline{\alpha}}A_{ca}\overline{A}_{\overline{\alpha}b} - \sum_{a,b,c,d=1}^n R_{a\overline{\alpha}b\overline{\alpha}}A_{ac}\overline{A}_{\overline{\alpha}d}$$

$$= CQB^{M_1}(A') + CQB^{M_2}(A'') + \sum_{i,j,\alpha} Ric_{i\overline{\alpha}}A_{ai}\overline{A}_{\overline{\alpha}j} + \sum_{a,\beta,i} Ric_{a\overline{\alpha}}A_{\alpha i}\overline{A}_{\beta j},$$

so the conclusion follows. Note that the positivity of CQB$_k$ implies that the dimension of the manifold must be at least 2. \Box

Since every irreducible compact Hermitian symmetric space with dimension bigger than one has positive CQB and $d^4$CQB by [18], the above corollary allows us to conclude that
Corollary 2.3. Every compact Hermitian symmetric has positive $^1CQB$ and nonnegative $CQB$, and it has positive $CQB$ if and only if it does not have any $\mathbb{P}^1$ factor.

If $(M^n, g)$ is a compact Kähler manifold with nonnegative Ricci curvature, then by the work of Campana, Demailly and Peternell [3], the universal cover $\tilde{M}$ of $M$ is holomorphically and isometrically the product $\mathbb{C}^k \times M_1 \times M_2$, where the first factor (if $k > 0$) is the flat de Rham factor, and $M_1$ is Calabi-Yau (simply connected with trivial canonical line bundle), while $M_2$ is rationally connected. Also, there exists a finite cover $M'$ of $M$, such that the Albanese map $\pi : M' \to \text{Alb}(M')$ is surjective and is a holomorphic and metric fiber bundle with fiber $M_1 \times M_2$. Here the bundle being metric means that any point in the base is contained in a neighborhood over which the bundle is isometric to the product of the fiber with the base neighborhood.

Now if $(M^n, g)$ is a compact Kähler manifold with $CQB_1 \geq 0$, then since it has nonnegative Ricci, the above structure theorem applies. We claim that the Calabi-Yau factor cannot occur in this case:

Theorem 2.4. Let $(M^n, g)$ be a compact Kähler manifold with $CQB_1 \geq 0$. Then a finite cover $M'$ of $M$ is a holomorphic and metric fiber bundle over its Albanese torus, with fiber being a rationally connected manifold. In particular, if $M$ has no flat de Rham factor, then it is rationally connected.

Proof. The goal is to rule out the Calabi-Yau factor, namely, to show that if $M_1$ is a simply-connected compact complex manifold with $c_1 = 0$, then it cannot admit any Kähler metric with $CQB_1 \geq 0$. To see this, notice that we have shown that $(n - 1)\text{Ric} \geq \text{Ric}^\perp \geq 0$. So if $\text{Ric}(X, \overline{X}) = 0$ for $X \in T'M_1$, then $\text{Ric}^\perp(X, \overline{X}) = 0$ and $R(X, \overline{X}, X, \overline{X}) = 0$. Let $\eta$ be the Ricci (1,1)-form of $M_1$, then by

$$c_1 \cdot [\omega]^{n-1} = \int_{M_1} \eta \wedge \omega^{n-1} = \frac{1}{n} \int_{M_1} S \omega^n,$$

where $\omega$ is the Kähler form and $S$ the scalar curvature, we see that the vanishing of the first Chern class $c_1$ plus the nonnegativity of Ricci imply that $M_1$ has to be scalar flat hence Ricci flat. So the holomorphic sectional curvature is identically zero, contradicting the fact that $M_1$ is simply connected.

In fact for any pair of $X$ and $Y$ by choosing $\{E_i\}$ such that $E_1 = \frac{X}{|X|^2}$, and letting $A$ be the map with $A(\overline{E_1}) = Y$, $A(\overline{E_i}) = 0$ for $i \geq 2$ the argument above implies the following corollary.

Corollary 2.5. The assumption $CQB_1 \geq 0$ is equivalent to that for any $X$ and $Y$,

$$|X|^2 \text{Ric}(Y, \overline{Y}) - R(X, \overline{X}, Y, \overline{Y}) \geq 0.$$  \hspace{1cm} (2.1)

If $CQB_1 > 0$, then the above holds as a strict inequality if $X, Y$ are nonzero.

Remark: It is not hard to see that under the CQB $\geq 0$ assumption, any tangent vector $X \in T'M$ with $\text{Ric}(X, \overline{X}) = 0$ must be in the kernel of the curvature tensor $R$, namely, $R(X, \overline{Y}, Z, \overline{W}) = 0$ for any $Y, Z, W \in T'M$. 

Next, let us recall the notion of dual cross quadratic bisectional curvature \( (d\text{CQB}) \) introduced in [18]. It is a Hermitian quadratic form on linear maps \( A : T'M \rightarrow T''M \):

\[
d\text{CQB}(A) = \sum_{\alpha,\beta=1}^{n} R(A(E_{\alpha}), A(E_{\alpha}), E_{\beta}, E_{\beta}) + R(E_{\alpha}, E_{\beta}, A(E_{\alpha}), A(E_{\beta}))
\]  

(2.2)

where \( R \) again is the curvature tensor of \( M \) and \( \{ E_{\alpha} \} \) is a unitary frame of \( T'M \). The manifold \( (M^n, g) \) is said to have positive (or nonnegative) \( d\text{CQB} \), if at any point in \( M \), for any unitary frame \( E \) of \( T'M \) at \( p \), and for any non-trivial linear map \( A : T'M \rightarrow T''M \), the value \( d\text{CQB}_{E}(A) \) is positive (or nonnegative). Related to this there is a Ricci \( \text{curvature of } (X, \overline{X}) = \text{Ric}(X, \overline{X}) + H(X)/|X|^2 \).

It is proved in [18] that compact Kähler manifold \( M^n \) with positive Ricci \( \text{Ric}^+ > 0 \) is projective and simply connected. If \( d\text{CQB} > 0 \) it also satisfies \( H^1(M, T'M) = \{0\} \), so it is locally deformation rigid. Moreover \( d\text{CQB}_1 > 0 \) implies \( \text{Ric}^+ > 0 \). Strictly analogous to the nonnegative CQB case, we have the following

**Theorem 2.6.** A Kähler manifold with positive (or nonnegative) \( d\text{CQB}_1 > 0 \) will have positive (or nonnegative) Ricci. A compact Kähler manifold with nonnegative \( d\text{CQB}_1 \geq 0 \) and without flat de Rham factor is rationally connected. Moreover \( \text{Ric}(X, \overline{X}) \geq \frac{1}{n+1} \text{Ric}^+(X, \overline{X}) \). In fact \( d\text{CQB}_1 \geq 0 \) is equivalent to the estimate:

\[
|X|^2 \text{Ric}(Y, \overline{Y}) + R(X, \overline{X}, Y, \overline{Y}) \geq 0
\]

(2.3)

for any pair of \((1,0)\)-type vectors \(X, Y\). If \( d\text{CQB}_1 > 0 \), then the above holds as a strict inequality if \( X, Y \) are nonzero.

**Corollary 2.7.** Let \( M^n = M_1 \times M_2 \) be a product Kähler manifold. Then for any \( 1 \leq k \leq n \), \( M \) has positive (or nonnegative) \( d\text{CQB}_k \) if and only if both \( M_1 \) and \( M_2 \) have positive (or nonnegative) \( d\text{CQB}_k \).

As noted in [18], when \( (M^n, g) \) is Kähler-Einstein, the CQB or \( d\text{CQB} \) conditions are given by the eigenvalue information for the curvature operator \( Q \) introduced by Calabi-Vessentini [2] and Itoh [11], which is the adjoint operator from \( S^2(T'M) \) into itself, defined by

\[
(Q(X \cdot Y), Z \cdot \overline{W}) = R(X, Z, Y, \overline{W})
\]

for any \((1,0)\) tangent vectors \(X, Y, Z, W\) in \( T'M\), where \( X \cdot Y = \frac{1}{2}(X \otimes Y + Y \otimes X) \) and the induced metric on \( S^2(T'M) \) is given by

\[
(X \cdot Y, Z \cdot \overline{W}) = \frac{1}{2} \left( g(X, Z)g(Y, \overline{W}) + g(X, \overline{Z})g(Y, W) \right).
\]

If we denote by \( \mu \) the constant Ricci curvature of \( M \), and by \( \lambda_1, \lambda_N \) the smallest and largest eigenvalue of \( Q \), respectively, then

\[
d\text{CQB} > 0 \iff \mu > \lambda_N, \quad \text{and} \quad d\text{CQB}_1 > 0 \iff \lambda_1 > -\mu.
\]

In section 4, we shall examine the eigenvalue bounds for the simplest kind of Kähler C-spaces, namely, the Type \( A \) spaces, and check the sign for CQB and \( d\text{CQB} \).

3. Fanoness of the nonflat factor

In this section we study further the factor in the splitting provided by Theorem 2.4. If we assume that the manifold \( (M, g) \) in Theorem 2.4 is simply-connected we show that \( M \) is a Fano manifold. Precisely we have the following slightly stronger result.
Theorem 3.1. Assume that $(M, g)$ be a compact Kähler manifold with CQB$_1 \geq 0$ (or dCQB$_1 \geq 0$). Assume that the universal cover $M$ does not have a flat de Rham factor. Then $M$ must be Fano. In fact the Kähler-Ricci flow evolves the metric $g$ into a Kähler metric $g(t)_{t \in (0, \varepsilon)}$ with positive Ricci curvature for some $\varepsilon$.

Proof. Here we adapt an idea of Böhm-Wilking in [1] where the authors proved that the Ricci flow deformation of a metric with nonnegative sectional curvature of a compact manifold with finite fundamental group evolves the initial metric into one with positive Ricci curvature for some short time. The assumption on the fundamental group is to effectively rule out the flat de Rham factor in its universal cover. A dynamic version of Hamilton’s maximum principle (cf. §1 of [1], Chapter 10 of [6], as well as [17]) was employed. Since CQB$_1 \geq 0$ (or dCQB$_1 \geq 0$) is different from the sectional curvature being nonnegative, we need to construct a different collection of invariant time-dependent convex sets and prove the corresponding estimates to show that $\text{Ric}(g(t)) > 0$. We shall focus on the case CQB$_1 \geq 0$ since the other case is similar.

Let $g(t)$ be the solution to Kähler-Ricci flow with initial metric $g$ satisfying CQB$_1 \geq 0$:

$$\frac{\partial}{\partial t} g_{\alpha\beta}(t) = -R_{\alpha\beta}, \quad g(0) = g$$

where $R_{\alpha\beta}$ denoted the Ricci curvature of $g(t)$. By Hamilton’s maximum principle we can focus on the study of a collection of sets $\{ C(t) \}$, each being a convex subset of the space of algebraic curvature operators satisfying the following conditions:

$$0 \leq \text{Ric}(X, \overline{X}), \ \forall X \in T'_x M;$$

$$\left| \text{Ric}(X, \overline{Y}) - R^g_{X\overline{Y}ZZ}(t) \right|^2 \leq (D_1 + tE_1) \text{Ric}(X, \overline{X}) \cdot \text{Ric}(Y, \overline{Y}), \ \forall X, Y, Z, |Z| = 1;$$

$$\| R \| \leq D_2 + tE_2.$$

Here in (3.3) $R$ is viewed as the curvature operator and $\| \cdot \|$ is the natural norm extended to the corresponding tensors from the Kähler metric on $T'_x M$.

First we need to check that the sets $C(t)$ are convex. Clearly (3.1) and (3.3) are convex conditions. For (3.2) let $R$ and $S$ be two Kähler curvature operators. We shall check that if (3.2) holds for $R$ and $S$ then it holds for $\eta R + (1 - \eta)S$ for $\eta \in [0, 1]$. Given $Z$ with $|Z| = 1$, $\text{Ric}(X, \overline{Y}) - R_{X\overline{Y}ZZ}$ is a Hermitian symmetric form on $T'_x M$, which we denote it as $A$, and denote the corresponding one for the curvature operator $S$ as $B$. We also denote $R(X, \overline{X})$ and $R(Y, \overline{Y})$ as $a_1$ and $a_2$. Similarly we have $b_1$ and $b_2$ for the corresponding Ricci of the curvature operator $S$. Then

$$|\eta A + (1 - \eta)B|^2 = \eta^2|A|^2 + \eta(1 - \eta)(A\overline{B} + B\overline{A}) + (1 - \eta)^2|B|^2$$

$$\leq \eta^2|A|^2 + 2\eta(1 - \eta)|A||B| + (1 - \eta)^2|B|^2$$

$$\leq (D_1 + tE_1) \left( \eta^2a_1a_2 + 2\eta(1 - \eta)\sqrt{a_1a_2}b_1b_2 + (1 - \eta)^2b_1b_2 \right)$$

$$\leq (D_1 + tE_1) (\eta a_1 + (1 - \eta)b_1)(\eta a_2 + (1 - \eta)b_2).$$

This completes the proof of the convexity of $C(t)$. Recall that after applying the Uhlenbeck’s trick [8] the Kähler-Ricci flow evolves the curvature tensor $R$ by the following PDE:

$$\left( \frac{\partial}{\partial t} - \Delta \right) R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\rho\sigma} R_{\gamma\delta\rho\sigma} + R_{\alpha\delta\rho\sigma} R_{\gamma\beta\rho\sigma} - R_{\alpha\rho\gamma\delta} R_{\rho\beta\delta}. \quad (3.4)$$
Here computation is with respect to a unitary frame. Tracing it gives the evolution equation of the Ricci curvature:

$$\left( \frac{\partial}{\partial t} - \Delta \right) R_{\alpha\beta} = R_{\alpha\beta pq} R_{pq}. \tag{3.5}$$

We shall show that the set $C(t)$ defined by (3.1), (3.2) and (3.3) are invariant under the equation (3.4) and (3.5). Hamilton’s maximum principle (see §1 of [1]) allows us to drop the diffusion term in verifying the invariance.

We first show that (3.2) holds at $t = 0$ since by Theorem 2.1 we have that (3.1) holds at $t = 0$, and it is easy to choose $D_2$ and $E_2$ to make $\epsilon$ is sufficiently small.

By Theorem 2.1, in particular (2.1), we have that for any $Z$ with $|Z| = 1$, $A(X, \overline{Y}) - R_{X \overline{Y} ZZ} = 0$ is a Hermitian symmetric tensor which is nonnegative. Diagonalize $A$ with a unitary frame $\{E_i\}$ and eigenvalues $\{\lambda_i\}$. Then we compute that for $X = x^i E_i$ and $Y = y^j E_j$

$$|A_{ij} x^i \bar{y}^j|^2 = \sum \lambda_i |x^i|^2 \sum \lambda_j |y^j|^2 \leq (\text{Ric}(X, \bar{X}) - R_{XXZZ}) \cdot (\text{Ric}(Y, \bar{Y}) - R_{YYZZ}) \leq \sum_{i=1}^n (\text{Ric}(X, \bar{X}) - R_{XXE_i E_i}) \sum_{j=1}^n (\text{Ric}(Y, \bar{Y}) - R_{YYE_i E_j}) \leq (n-1)^2 \text{Ric}(X, \overline{X}) \text{Ric}(Y, \overline{Y}).$$

Here $\{E_j\}$ is another unitary frame so chosen that $E_1 = Z$. Hence if we choose $D_1 = (n-1)^2$ the estimate (3.2) holds at $t = 0$.

Now we need to verify that the PDE/ODE preserves the set $C(t)$. For that we only need to prove that the time derivative of the convex condition lies inside the tangent cone of the convex set. The trick of [1] is to chose $D_1$ sufficiently large (compared with $D_1, D_2, E_2$) to make sure that (3.2) stay invariant under the PDE (3.4) (or the corresponding ODE $\frac{\partial}{\partial t} \text{Rm} = \text{Rm}^2 + \text{Rm}^\#$) for $t \in [0, \epsilon]$ if $\epsilon$ is very small. With a suitably chosen $D_2$, it is easy to have (3.3). In fact we may choose $E_2 = 1$ if $\epsilon$ is small. For (3.1), if $\text{Ric}(X, \bar{X})$ ever becomes zero for some $X$, then within $C(t)$ by (3.2), we have $\text{Ric}(X, \overline{Y}) - R_{X \overline{Y} ZZ} = 0, \forall Y, Z$.

This then via the polarization implies that $R_{X \overline{Y} ZZ} = 0, \forall Y, Z, W$. Thus (3.5) implies $\frac{\partial}{\partial t} \text{Ric}(X, \overline{X}) \geq R_{X \overline{X} pq} R_{pq} = 0$. This shows that (3.1) is preserved by (3.5).

As in [1], the main issue is to show that (3.2) is preserved under the flow, namely (3.4) and (3.5). For this it suffices to show that as long as $R$ is in $C(t)$,

$$\frac{\partial}{\partial t} \left( (D_1 + tE_1) \text{Ric}(X, \overline{X}) \cdot \text{Ric}(Y, \overline{Y}) - \text{Ric}(X, \overline{Y}) - R_{X \overline{Y} ZZ} \right)^2 \geq 0. \tag{3.6}$$

Direct calculation shows that the left hand side of the above inequality is

$$E_1 \text{Ric}(X, \overline{X}) \cdot \text{Ric}(Y, \overline{Y}) + (D_1 + tE_1) \left( R_{XXpq} \text{Ric}(Y, \overline{Y}) + R_{YYpq} \text{Ric}(X, \overline{X}) \right) R_{pq} - 2R \left( \frac{\partial}{\partial t} \text{Ric}(X, \overline{Y}) - \frac{\partial}{\partial t} R_{X \overline{Y} ZZ} \right) \left( \text{Ric}(X, \overline{Y}) - R_{X \overline{Y} ZZ} \right).$$

We shall show that for $\epsilon$ small and $t \in [0, \epsilon]$ the above is nonnegative. Namely the first term dominates the rests. By (3.2), by letting $\epsilon \leq \frac{1}{E_1}$ (with $E_1$ to be decided later),
\((D_1 + tE_1) \leq 2D_1\). In the mean time \(E_1\) is chosen to be large comparing with \(D_1^2D_2\). First (3.2), together with \(|\text{Ric}(X, Y)\| \leq \sqrt{\text{Ric}(X, X)\text{Ric}(Y, Y)}\), imply that

\[
|R_{X\overline{Y}\overline{Z}\overline{W}}| \leq 4D_1\sqrt{\text{Ric}(X, X)\text{Ric}(Y, Y)}
\]

which then implies that

\[
|R_{X\overline{Y}pq}\overline{R}_{q\overline{p}}| \leq 4nD_1D_2\text{Ric}(X, X).
\]

This, together with \(tE_1 \leq 1\), imply that

\[
(D_1 + tE_1) \left(R_{X\overline{Y}pq}\text{Ric}(Y, Y) + R_{Y\overline{Z}pq}\text{Ric}(X, X)\right) \overline{R}_{q\overline{p}} \geq -16nD_1^2D_2\text{Ric}(X, X) \cdot \text{Ric}(Y, Y).
\]

To handle the term involving \(\frac{\partial}{\partial t}R_{X\overline{Y}\overline{Z}\overline{W}}\) we observe the following estimates:

\[
|R_{X\overline{Y}\overline{Z}\overline{W}}| \leq 32nD_1\sqrt{nD_2}\sqrt{\text{Ric}(X, X)}
\]

\[
|R_{Y\overline{Z}\overline{Y}\overline{W}}| \leq 32nD_1\sqrt{nD_2}\sqrt{\text{Ric}(Y, Y)}, \quad \forall \, U, Z, W, |U| = |Z| = |W| = 1.
\]

These can be derived easily out of (3.7) and (3.3). Now note that

\[
\left|\left(\text{Ric}(X, Y) - R_{X\overline{Y}\overline{Z}\overline{W}}\right)\right| \leq 2D_1\sqrt{\text{Ric}(X, X)\text{Ric}(Y, Y)}.
\]

Hence we only need to establish that

\[
\left|\left(\frac{\partial}{\partial t}\text{Ric}(X, Y) - \frac{\partial}{\partial t}R_{X\overline{Y}\overline{Z}\overline{W}}\right)\right| \leq C(D_1, D_2, n)\sqrt{\text{Ric}(X, X)\text{Ric}(Y, Y)}
\]

for some positive \(C\) depends on \(D_1, D_2\) and \(n\). By (3.4) and (3.5) we have that

\[
\left(\frac{\partial}{\partial t}\text{Ric}(X, Y) - \frac{\partial}{\partial t}R_{X\overline{Y}\overline{Z}\overline{W}}\right) = R_{X\overline{Y}\overline{Z}\overline{W}}\text{Ric}_{Z\overline{W}} - R_{Z\overline{W}pq}\overline{R}_{q\overline{p}X}\overline{T} - R_{Z\overline{W}pq}\overline{R}_{q\overline{p}X}\overline{T} + R_{Z\overline{W}pq}\overline{R}_{q\overline{p}X}\overline{T}.
\]

Putting Estimates (3.7), (3.9) and (3.10) together we have the estimate we want. Taking \(E_1 \geq 100C(D_1, D_2, n)\) we have proved (3.6). Hence \(\{C(t)\}\) is an invariant collection of convex subsets under the Kähler-Ricci flow.

If \(\text{Ric}(g(t))\) has a nontrivial kernel, the strong maximum principle (see for example, pages 675-676 of [1]) takes effect to imply that the universal cover splits a factor according to the distribution provided by the vectors in the kernel of the Ricci curvature. The factor is flat since by (3.2) the kernel of \(\text{Ric}\) would be the kernel of the curvature tensor. This is contradictory to our assumption that the universal cover does not contains a line. Thus we have proved that \(\text{Ric}(g(t)) > 0\) for any \(t \in (0, \epsilon)\).

An argument similar as [1] was also employed by Liu in [13] to the non-positive setting to conclude that the deformed metric has negative Ricci curvature if the initial metric has non-positive bisectional curvature.
4. **Kähler C-spaces**

Recall that Kähler C-spaces are the orbit spaces of the adjoint representation of compact semisimple Lie groups. Any such space is the product of simple Kähler C-spaces, and all simple Kähler C-spaces can be obtained in the following way.

Let \( \mathfrak{g} \) be a simple complex Lie algebra. They are classified as the four classical sequences \( A_r = \mathfrak{sl}_{r+1} \)( \( r \geq 1 \)), \( B_r = \mathfrak{so}_{2r+1} \)( \( r \geq 2 \)), \( C_r = \mathfrak{sp}_{2r} \)( \( r \geq 3 \)), \( D_r = \mathfrak{so}_{2r} \)( \( r \geq 4 \)) and the exceptional ones \( E_6, E_7, E_8, F_4 \) and \( G_2 \).

Let \( \mathfrak{h} \subset \mathfrak{g} \) be a Cartan subalgebra with corresponding root system \( \Delta \subset \mathfrak{h}^* \). So we have \( \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathbb{C} E_\alpha \) where \( E_\alpha \) is a root vector of \( \alpha \). Let \( r = \text{dim}_\mathbb{C} \mathfrak{h} \) and fix a fundamental root system \( \{ \alpha_1, \ldots, \alpha_r \} \). This gives an ordering in \( \Delta \), and let \( \Delta^+, \Delta^- \) be the set of positive or negative roots. Each \( \beta \in \Delta^+ \) can be expressed as \( \beta = \sum_{i=1}^r n_i(\beta) \alpha_i \). For a fixed nonempty subset \( \Phi \subset \{ \alpha_1, \ldots, \alpha_r \} \), denote by

\[
\Delta^+_{\Phi} = \{ \beta \in \Delta^+ \mid n_i(\beta) > 0 \text{ for some } \alpha_i \in \Phi \}.
\]

Let \( G \) be the simple complex Lie group with Lie algebra \( \mathfrak{g} \) and \( L \) the closed subgroup with Lie subalgebra \( \mathfrak{h} = \mathfrak{g} \oplus \bigoplus_{\alpha \in \Delta} \mathbb{C} E_\alpha \). Then \( M^n = G/L \) is a simple Kähler C-space, and all simple Kähler C-spaces can be obtained in this way. The complex dimension \( n \) of \( M \) is equal to the cardinality \( |\Delta^+| \), while \( b_2(M) = |\Phi| \). The tangent space \( TM \) at the point \( eL \) can be identified with the subspace \( \mathfrak{m}^+ = \bigoplus_{\beta \in \Delta^+_{\Phi}} \mathbb{C} E_\beta \) of \( \mathfrak{g} \).

Following Itoh [11], we will denote this simple Kähler C-space as \( M^n = (\mathfrak{g}, \Phi) \).

Next let us recall the Chevalley basis (see [9] or Prop. 11 of [14]). Let \( B \) be the Killing form of \( \mathfrak{g} \). For each \( \alpha \in \Delta \), let \( H_\alpha \) be the unique element in \( \mathfrak{h} \) such that \( B(H_\alpha, H) = \alpha(H) \) for any \( H \in \mathfrak{h} \). One can always choose root vectors \( E_\alpha \) of \( \mathfrak{g}_0 \), so that \( \{ E_\alpha, E_{-\alpha} \} = H_\alpha, E_\alpha = -E_{-\alpha} \), and \( N_{\alpha, -\alpha} = -N_{\alpha, \beta} \), where \( N_{\alpha, \beta} \) is defined by \( [E_\alpha, E_\beta] = N_{\alpha, \beta} E_{\alpha + \beta} \) for any \( \alpha, \beta \in \Delta \) with \( \alpha \neq -\beta \). When \( \alpha + \beta \) is not a root, then \( N_{\alpha, \beta} = 0 \).

Denote by \( z_\alpha = B(E_\alpha, E_{-\alpha}) \). Then \( [E_\alpha, E_{-\alpha}] = z_\alpha H_\alpha \), and \( z_\alpha \) are all real and \( z_{-\alpha} = z_\alpha \) for each \( \alpha \). Now we describe the invariant Kähler metrics on \( M \). Such a metric \( g \) makes the tangent frame \( F := \{ E_\alpha, \alpha \in \mathfrak{m}^+ \} \) an orthogonal frame, with \( g(E_\alpha, E_\beta) = g_\alpha z_\alpha \) where \( g_\alpha \) satisfy the following additive condition with respect to \( \Phi \):

Write \( \Phi = \{ \alpha_{i_1}, \ldots, \alpha_{i_m} \} \), where \( 1 \leq i_1 < \cdots < i_m \leq r \). Assign \( g_{\alpha_{i_j}} = c_j > 0 \) arbitrarily, and require \( g_{\beta} = n_{\alpha_{i_j}}(\beta)c_1 \cdots + n_{\alpha_{i_m}}(\beta)c_m \) for any \( \beta = n_{\alpha_{i_j}}(\beta)\alpha_1 + \cdots + n_{\alpha_{i_m}}(\beta)\alpha_r \in \Delta^+_\Phi \).

Denote this metric as \( g = g(c_1, \ldots, c_m) \). So the invariant Kähler metrics on \( M \) are determined by \( m = b_2 \) positive constants \( c_1, \ldots, c_m \). It turns out (see §3.2 of [14]) that the metric is Einstein if and only if up to scaling, \( g_\alpha = \sum_{\beta \in \Delta^+_{\Phi}} B(\alpha, \beta) \) for any \( \alpha \in \Delta^+_{\Phi} \).

Following the computation initiated in [11], Lohove ([14], Prop 16) completed the curvature formula (for \( M^n, g \)) under the Chevalley frame \( F \), which we will describe below. For \( \alpha, \beta, \gamma, \delta \in \Delta^+_{\Phi} \), write \( R(E_\alpha, E_\beta, E_\gamma, E_\delta) = R_{\alpha, \beta, \gamma, \delta} \). A highly distinctive property of the curvature of \( M \) is that

\[
R_{\alpha, \beta, \gamma, \delta} = 0 \quad \text{unless} \quad \alpha + \gamma = \beta + \delta. \tag{4.1}
\]

To take advantage of the symmetry of curvature for Kähler metrics, let us consider the order relation \( \prec \) in \( \Delta \): for \( \alpha \neq \beta \in \Delta \), write \( \alpha < \beta \) if \( n_s(\alpha) < n_s(\beta) \) but \( n_1(\alpha) = n_1(\beta) \) for all \( 1 \leq i < s \) (if \( s > 1 \)).

For \( R_{\alpha, \beta, \gamma, \delta} \) with \( \alpha + \gamma = \beta + \delta \), by Kähler symmetries, we may assume that \( \alpha \) is the smallest, and \( \beta \leq \delta \). If \( \alpha = \beta \), then \( \gamma = \delta \), so we are left with \( R_{\alpha, \gamma, \gamma} \) where \( \alpha \leq \gamma \). If \( \alpha \neq \beta \), then
we are left with the case $\alpha < \beta \leq \delta < \gamma$. In the first case, Lohove obtained that, for any $\alpha$, $\gamma \in \Delta^+_\Phi$ with $\alpha \leq \gamma$:

$$
R_{\alpha \gamma} = \begin{cases}
 g_\alpha z_\alpha z_\gamma B(H_\alpha, H_\gamma) + \frac{g_\alpha g_\gamma}{g_{\alpha+\gamma}}z_{\alpha+\gamma}N_{\alpha+\gamma}^2, & \text{if } \gamma - \alpha \in \Delta^+_\Phi; \\
 g_\gamma z_\alpha z_\gamma B(H_\alpha, H_\gamma) + \frac{g_\alpha}{g_{\alpha+\gamma}}z_{\alpha+\gamma}N_{\alpha+\gamma}^2, & \text{if } \gamma - \alpha \notin \Delta^+_\Phi.
\end{cases}
$$

(4.2)

For the second case, he obtained that, for any $\alpha, \beta, \gamma, \delta \in \Delta^+_\Phi$ with $\alpha < \beta \leq \gamma$ and $\alpha + \gamma = \beta + \delta$,

$$
R_{\alpha \beta \gamma} = \begin{cases}
 g_\alpha z_\alpha - \beta N_{\alpha - \beta}N_{\gamma - \delta} + \frac{g_\alpha g_\delta}{g_{\alpha+\gamma}}z_{\alpha+\gamma}N_{\alpha+\gamma}N_{\beta, \delta}, & \text{if } \gamma - \beta \in \Delta^+_\Phi; \\
 g_\delta z_\alpha - \beta N_{\alpha - \beta}N_{\gamma - \delta} + \frac{g_\alpha g_\delta}{g_{\alpha+\gamma}}z_{\alpha+\gamma}N_{\alpha+\gamma}N_{\beta, \delta}, & \text{if } \gamma - \beta \notin \Delta^+_\Phi.
\end{cases}
$$

(4.3)

Note that in [14] the curvature $R$ differs from here by a minus sign, as he is using a different sign convention. Next let us specialize to the simplest case, namely, when

$$
g = A_r = \mathfrak{sl}(r+1)
$$

is the space of all traceless complex $(r+1)$ square matrices. A Cartan subalgebra $\mathfrak{h}$ is given by all (traceless) diagonal matrices. The Killing form $B$ is $B(X,Y) = \text{tr}(XY)$. The root system is given by $\Delta = \{\alpha_{ij} : 1 \leq i, j \leq r + 1\}$, where $\alpha_{ij}(H) = H_{ii} - H_{jj}$ for any $H \in \mathfrak{h}$, with a fundamental basis $\{\alpha_1, \ldots, \alpha_r\}$ where $\alpha_i = \alpha_{i(i+1)}$. The positive roots are $\Delta^+ = \{\alpha_{ij} : 1 \leq i < j \leq r + 1\}$, with $-\alpha_{ij} = \alpha_{ji}$.

Denote by $E_{ij}$ the $(r+1)\times(r+1)$ matrix whose only nonzero entry is 1 at the $(i, j)$-th position, and write $H_{ij} = E_{ii} - E_{jj}$. Then $\{H_{ij}, E_{ij}\}$ forms a Chevalley basis. Since $[E_{ij}, E_{ij}] = H_{ij}$, we know that $z_\alpha = 1$ for all $\alpha \in \Delta$. Thus the square norm $g(E_\alpha, \overline{E}_\alpha) = g_\alpha$. To simplify our further discussions, let us introduce the following notations. For any $\alpha < \gamma$ in $\Delta^+$, we will denote by

$$
\gamma \equiv \alpha \iff \gamma = \alpha_{ij}, \ \alpha = \alpha_{j} \quad \text{for some } 1 \leq i < j < k \leq r + 1
$$

$$
\gamma \equiv' \alpha \iff \gamma = \alpha_{ik}, \ \alpha = \alpha_{i} \quad \text{for some } 1 \leq i < j < k \leq r + 1
$$

$$
\gamma \equiv'' \alpha \iff \gamma = \alpha_{ik}, \ \alpha = \alpha_{j} \quad \text{for some } 1 \leq i < j < k \leq r + 1
$$

$$
\gamma \equiv \alpha \iff \gamma \equiv' \alpha \quad \text{or} \quad \gamma \equiv'' \alpha
$$

Since $B(H_{ij}, H_{kl}) = \text{tr}((E_{ii} - E_{jj})(E_{kk} - E_{kk})) = \delta_{ik} + \delta_{ji} - \delta_{il} - \delta_{jk}$, we get $B(H_\alpha, H_\alpha) = 2$ for each $\alpha$, and for any $\alpha < \gamma$ in $\Delta^+$, we have

$$
B(H_\alpha, H_\gamma) = \begin{cases}
 -1, & \text{if } \gamma \equiv \alpha \\
 1, & \text{if } \gamma \equiv \alpha \\
 0, & \text{otherwise}
\end{cases}
$$

Also, since $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$, we get that for any $\alpha < \gamma$ in $\Delta^+$,

$$
N_{\alpha, \gamma} = \begin{cases}
 -1, & \text{if } \gamma \equiv \alpha \\
 0, & \text{otherwise}
\end{cases}
$$

Also, for any $\alpha < \beta \in \Delta^+$,

$$
N_{\alpha, -\beta} = \begin{cases}
 -1, & \text{if } \gamma \equiv'' \alpha \\
 1, & \text{if } \gamma \equiv' \alpha \\
 0, & \text{otherwise}
\end{cases}
$$
Note that for $\delta < \gamma \in \Delta^+$, we have $N_{\gamma,-\delta} = -N_{\gamma,\delta} = N_{\delta,-\gamma}$. Putting all these info into the Itoh-Lahove curvature formula, we get $R_{\alpha\alpha\pi\pi} = 2g_{\alpha}$, and for any $\alpha < \gamma$ in $\Delta^+$,

$$R_{\alpha\alpha\pi\gamma} = \begin{cases} -\frac{g_{\alpha\gamma}}{g_{\alpha\gamma}}, & \text{if } \gamma \cap \alpha \\ \frac{g_{\alpha\gamma}}{g_{\alpha\gamma}}, & \text{if } \gamma \subset \alpha \\ 0, & \text{otherwise} \end{cases}$$

Also, for $\alpha < \beta < \delta < \gamma$ in $\Delta^+$ with $\alpha + \gamma = \beta + \delta$, only two cases will result in nonzero values for $R_{\alpha\beta\gamma\delta}$, namely, either when $\gamma \cup \alpha = \delta \cup \beta$ and $\gamma^\prime \delta$, or when $\beta \cup \alpha$, $\delta \cup \gamma$, and $\gamma = \beta + \delta - \alpha$. In the first case the curvature equals to $-\frac{2g_{\alpha\delta}}{g_{\alpha\gamma}}$, and in the second case the curvature equals to $g_{\alpha}$. Note that these two cases can be described equivalently as: there exist 1 $\leq i < p < q < k \leq r + 1$ such that $\gamma = \alpha_{ip}$, $\beta = \alpha_{pq}$, $\gamma = \alpha_{iq}$, $\alpha = \alpha_{pk}$ for the first case, while $\delta = \alpha_{ip}$, $\beta = \alpha_{pk}$, $\gamma = \alpha_{ik}$, $\alpha = \alpha_{pq}$ for the second case.

Now let us switch to the unitary frame $\tilde{E}_{\alpha} = \frac{E_{\alpha}}{\sqrt{g_{\alpha}}}$ of $m^+$. For the sake of convenience, we will still use $R_{\alpha\beta\gamma\delta}$ to denote the curvature component $R(\tilde{E}_{\alpha}, \tilde{E}_{\beta}, \tilde{E}_{\gamma}, \tilde{E}_{\delta})$. Also, to avoid clumsy notations, we will write $g_{ab}$ simply as $g_{ab}$. Up to the Kähler symmetries, the only non-zero components of the curvature are

$$R_{\alpha\alpha\pi\pi} = \frac{2}{g_{\alpha}}, \quad \alpha \in \Delta^+$$

(4.4)

$$R_{\alpha\beta\gamma\gamma} = \begin{cases} -\frac{1}{g_{\beta}}, & \text{if } \exists i < j < k : \gamma = \alpha_{ij}, \quad \alpha = \alpha_{jk} \\ \frac{1}{g_{\beta}}, & \text{if } \exists i < j < k : \gamma = \alpha_{ik}, \quad \alpha = \alpha_{ij} \text{ or } \alpha_{jk} \end{cases}$$

(4.5)

where we assumed $\alpha < \gamma$. For $\alpha < \beta < \delta < \gamma$ in $\Delta^+$, the curvature component $R_{\alpha\beta\gamma\delta}$ will be equal to the following non-zero values only when there are 1 $\leq i < p < q < k \leq r + 1$ such that

$$R_{\alpha\beta\gamma\delta} = \begin{cases} -\frac{\sqrt{g_{\alpha\beta}}}{g_{\beta}} \frac{g_{\gamma}}{\sqrt{g_{\gamma}}}, & \text{if } \delta = \alpha_{ip}, \beta = \alpha_{pk}, \gamma = \alpha_{iq}, \alpha = \alpha_{pq} \\ \frac{\sqrt{g_{\alpha\beta}}}{g_{\beta}} \frac{g_{\gamma}}{\sqrt{g_{\gamma}}}, & \text{if } \delta = \alpha_{iq}, \beta = \alpha_{pk}, \gamma = \alpha_{ik}, \alpha = \alpha_{pq} \end{cases}$$

(4.6)

Now check the sign for CQB or $d$CQB. First let us consider the case when $\Phi = \{\alpha_1, \ldots, \alpha_r\}$, namely, when $M^n = SU(r + 1)/T$ is the flag manifold, where $T$ is a maximal torus. We have $n = \frac{1}{2}r(r + 1)$, $b_2 = r$, and $\Delta^+ = \Delta^+$. We will choose $g$ to be the Kähler-Einstein metric. In this case, all $c_{ij} = 1$, and $g_{ab} = k - i$. It is easy to see that the Ricci curvature is constantly equal to $\mu = 2$.

For any symmetric $n \times n$ matrix $A$, the quadratic form $\langle Q(A), \overline{\alpha} \rangle = \sum_{a,b,c,d=1}^n R_{abcd} A_{ac} \overline{A}_{bd}$ is equals to

$$\sum_{a} R_{\alpha\alpha\pi\pi} |A_{\alpha a}|^2 + \sum_{\alpha < \gamma} 4R_{\alpha\beta\gamma\gamma} |A_{\alpha \gamma}|^2 + \sum_{\alpha < \beta < \gamma} 8\Re\{R_{\alpha\beta\gamma\delta} A_{\alpha \gamma} \overline{A}_{\beta \delta}\}

= \sum_{a} \frac{2}{g_a} |A_{\alpha a}|^2 + \sum_{i < j < k} \frac{4}{g_{ij}} (|A_{ij,ik}|^2 + |A_{jk,ik}|^2 - |A_{jk,ij}|^2) + \sum_{i < p < q < k} 8\Re\{-\frac{\sqrt{g_p g_k}}{g_{ik}} A_{pk,ip} \overline{A}_{pk,ip} + \frac{\sqrt{g_p g_k}}{g_{ip} g_{pq}} A_{pq,ik} \overline{A}_{pk,ip}\}$$

(4.7)

Let us denote by $X$ and $Y$ the two terms in the last line above. We have

$$CQB_E(A) = \mu ||A||^2 - \langle Q(A), \overline{\alpha} \rangle, \quad dCQB_E(A) = \mu ||A||^2 + \langle Q(A), \overline{\alpha} \rangle.$$
In order to check that CQB $\geq 0$ and $d$CQB $> 0$ for $(SU(r + 1)/T, g)$, the flag manifold of type $A$ with Einstein metric, it suffices to take care of the two crossing terms $X$ and $Y$. For $Y$, the square root part of the coefficient is less than $\frac{1}{2}$, so we have

$$|Y| \leq \sum_{i<p<q<k} 4|A_{pq,ik}A_{pk,iq}| \leq \sum_{i<p<q<k} 2|A_{pq,ik}|^2 + 2|A_{pk,iq}|^2$$

Note that in $2|A|^2 = 2(A, \mathcal{A}) = \sum_\alpha |A_{\alpha\alpha}|^2 + 4 \sum_{\alpha<\beta} |A_{\alpha\beta}|^2$, each $|A_{pq,ik}|^2$ term or $|A_{pk,iq}|^2$ term will appear 4 times, so the $Y$ term will be dominated by $\mu||A||^2$ from above or below. For the $X$ term, let us fix $i < k$ with $k - i = t + 1 \geq 2$. Write $A_{ip,pk} = Z_p$, and write $p' = p - i$. Since the square root part of the coefficient of $X$ is less than 1, we have

$$|X| \leq \sum_{i<k} \sum_{1 \leq p' < q' \leq t} \frac{4}{t+1} (|Z_p|^2 + |Z_q|^2) = \sum_{i<p<k} \frac{4(t-1)}{t+1} |Z_p|^2.$$ 

Again since for each $i < j < k$, the term $|A_{ij,jk}|^2 = |Z_j|^2$ will appear 4 times in $\mu||A||^2$, the $X$ term will be dominated by $\mu||A||^2$ from above and below. Note that for the lower bound part, the term $|Z_p|^2$ will also emerge from the bisectional curvature terms, with coefficient $-\frac{4}{t+1}$. We have $-\frac{4(t-1)}{t+1} - \frac{4}{t+1} = -\frac{4t}{t+1} > -4$, so $d$CQB will be nonnegative, and actually positive since its vanishing would imply $A = 0$. We have thus proved Theorem 1.6 stated in the introduction.

Note that if $A$ has only non-trivial entries along the diagonal line for the simple roots, then $(Q(A), \mathcal{A}) = 2||A||^2$, so CQB is only nonnegative and not positive.

Next let us give a non-symmetric example of irreducible Kähler C-space with $b_2 > 1$ that has positive CQB. The smallest dimensional Type $A$ space which is non-symmetric and not a $\mathbb{P}^1$ bundle would be $M^{12} = SU(6)/SU(2) \times SU(2)$, or equivalently, $(A_5, \Phi) = (\mathbb{C}_6, \Phi)$ where $\Phi = \{\alpha_2, \alpha_4\}$. It has $n = 12$ and $b_2 = 2$. We have

$$\Delta^+_6 = \{\alpha_{kl} | 1 \leq k < l \leq 6\} \setminus \{\alpha_{12}, \alpha_{34}, \alpha_{56}\}.$$ 

Up to a scaling, the Kähler-Einstein metric $g$ has components $g_{kl} = g_{\alpha\alpha} = \sum_{\beta \in \Delta^+_6} B(\alpha_{kl}, \beta)$, so we have

$$g_{13} = g_{14} = g_{23} = g_{24} = 2, $$

$$g_{35} = g_{36} = g_{45} = g_{46} = 2, $$

$$g_{15} = g_{16} = g_{25} = g_{26} = 4.$$ 

Let us denote by $\Delta_1 = \{\alpha_{15}, \alpha_{16}, \alpha_{25}, \alpha_{26}\}$ and $\Delta_2 = \Delta^+_6 \setminus \Delta_1$. So the curvature components are $R_{\alpha\gamma\beta\gamma} = \frac{2}{3}$, which is $\frac{2}{3}$ for $\alpha \in \Delta_1$ and 1 for $\alpha \in \Delta_2$. While $R_{\alpha\gamma\beta\gamma}$ are given by (4.5). It is easy to see that the Ricci curvature is constantly $\mu = 2$ in this case. The crossing terms $R_{\alpha\gamma\beta\gamma}$ are given by (4.6). We have

$$\mu||A||^2 = \sum_\alpha 2|A_{\alpha\alpha}|^2 + \sum_{\alpha < \gamma} 4|A_{\alpha\gamma}|^2.$$ 

Now consider the quadratic form $(Q(A), \mathcal{A})$ given by (4.7). Let us examine the two terms $X$ and $Y$ in the last line of (4.7). For the term $Y$, note that $i < p < q < k$ could be from $(1, 2, 3, 4), (3, 4, 5, 6)$, in which case the square root part of the coefficient is $\frac{1}{2}$, or from $(1, 2, 3, 5), (1, 2, 3, 6), (1, 2, 4, 5), (1, 2, 4, 6), (1, 2, 5, 6), (1, 3, 5, 6), (1, 4, 5, 6), (2, 3, 5, 6), or...
Assume that the universal cover $\tilde{M}$ does not have a flat de Rham factor. Then $M$ must admit a metric with Ric $< 0$. In fact the Kähler-Ricci flow evolves the metric $g$ into a Kähler metric $g(t)_{t \in (0, \epsilon)}$ with negative Ricci curvature for some $\epsilon$.
Next construct examples of compact Kähler manifolds with negative (non-positive) CQB and $d\text{CQB}$. First of all, if $M^n$ is a compact quotient of a Hermitian symmetric space $M$ of non-compact type, then by [2], we see that $M$ always has $\text{CQB} < 0$ and $\text{dCQB} \leq 0$, and it will have $\text{CQB} < 0$ when and only when $\bar{M}$ does not have the unit disc as an irreducible factor.

For non-locally Hermitian symmetric examples, we adapt the construction of strongly negatively curved manifolds by Mostow and Siu [16] and by the second named author [24], [25]. To state the result, let us recall the notion of good coverings.

A finite branched cover $f : M^n \to N^n$ between two compact complex manifolds is called a good cover, if for any $p \in M$, there exists locally holomorphic coordinates $(z_1, \ldots, z_n)$ centered at $p$ and $(w_1, \ldots, w_n)$ centered at $f(p)$, such that $f$ is given by $w_i = z_i^{m_i}$, $1 \leq i \leq n$, where $m_i$ are positive integers. Note that the branching locus $B$ and ramification locus $R$ are necessarily normal crossing divisors in this case.

In [16], Mostow and Siu computed the curvature for the Bergman metric of the Thullen domain $\{ |z_1|^{2m} + |z_2|^2 < 1 \}$, and used it to construct examples of strongly negatively curved surfaces which is not covered by ball. In [24], the second named author generalized this to higher dimensions, and also at the quotient space level using the Poincaré distance, and showed that (see Theorem 1 of [24]) if $N$ is a compact smooth quotient of the ball, and $B \subset N$ a smooth totally geodesic divisor (possibly disconnected), then for any good cover $f : M \to N$ branched along $B$, $M$ admits a Kähler metric with negative complex curvature operator. We will use this computation to claim the following:

**Theorem 5.3.** Let $N^n$ ($n \geq 2$) be a smooth compact quotient of the ball, equipped with the complex hyperbolic metric, and let $B \subset N$ be a smooth totally geodesic divisor (possibly disconnected). If $f : M \to N$ is a good cover branched along $B$, then $M$ admits a Kähler metric with negative $\text{CQB}$ and negative $d\text{CQB}$.

**Remark:** Such a manifold $M$ is not homotopy equivalent to any locally Hermitian symmetric space, and it is strongly rigid in the sense of Siu, namely, any compact Kähler manifold homotopy equivalent to $M$ must be (anti)biholomorphic to $M$.

**Proof.** The construction of the Kähler metrics $\omega_\varepsilon$ is exactly the same as in the proof of Theorem 1 of [24]. Notice that at the point $p$ in a tubular neighborhood $V$ of the ramification locus $R$, there exists tangent frame $e$ at $p$ such that $e_i \perp e_j$ whenever $i \neq j$, and under $e$ the only non-zero curvature components of $\omega_\varepsilon$ are $R^{\overline{i}j\overline{j}}$, with

$$-R^{\overline{i}j\overline{j}} = b, \quad -R^{\overline{i}i\overline{i}} = c, \quad -R^{\overline{i}i\overline{j}} = 2e, \quad -R^{\overline{j}j\overline{j}} = e$$

for any $2 \leq i < j$. It was shown that $b > 0$, $c > 0$, $e > 0$, and $nbc > (n-1)^2 e^2$.

Note that if we normalize $e$, namely, replace $e_k$ by $\frac{e_k}{R^{\overline{k}k}}$ for each $k$, then the above inequalities on $b$, $c$, and $e$ still holds. So let us assume that $e$ is unitary at $p$. For any non-trivial $n \times n$
matrix $A$, we have $-\text{CQB}_e(A) = P - Q$, and $-\text{dCQB}_e(A) = P + Q$, where

$$
P &= -\sum_{i,j,k,\ell} R_{ij\ell k} A_{ij} A_{\ell k} = -\sum_{i,k,\ell} R_{i\ell k} |A_{i\ell}|^2
$$

$$
Q &= -\sum_{i,j,k,\ell} R_{ij\ell k} \overline{A}_{ij} \overline{A}_{\ell k} = -\sum_{i,k} R_{i\ell} |A_{i\ell}|^2 - \sum_{i < k} R_{i\ell k} |A_{ik} + A_{ki}|^2
$$

Clearly, $P + Q > 0$ for all $A \neq 0$, and if we write $t_{ij} = |A_{ij}|^2$, we have

$$
P - Q = (n-1) c t_{11} + (c + (n-2) e) \sum_{i,k > 1} t_{ik} + 
$$

$$
\sum_{i > 1} \left( (b + (n-2) c) t_{1i} + ne t_{i1} - 2c \Re(e_{1i} \overline{A}_{1i}) \right),
$$

which is positive as $n bc > (n-1) c^2 > e^2$. So the metric $\omega_e$ has CQB < 0 and $d\text{CQB} < 0$ in $V$, for any $\varepsilon > 0$. By choosing $\varepsilon$ sufficiently small, one see that CQB and $d\text{CQB}$ will be negative everywhere in $M$.

By [24] and [25], we see that there are many examples of such $M$ in $n = 2$. An example in $n = 3$ was constructed by M. Deraux in [7], and we are not aware of any higher dimensional such constructions, even though it has been widely believed that there should be plenty in all dimensions.

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References


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