

1. (40 pts.) Let \mathcal{R} be the region between the two parabolas $y = x^2$ and $x = y^2$.

Let \mathcal{V} be the volume obtained when \mathcal{R} is rotated about the y -axis.

- (a) Sketch the region \mathcal{R} . Include in your drawing the coordinates of the point where the parabolas intersect.

Ans. We omit the picture. The region lies in the first quadrant, is bounded below by $y = x^2$ and above by $x = y^2$, and the points of intersection are $(0,0)$ and $(1,1)$. You can find the intersection points from your sketch (and then easily check them in your head). Alternatively, you can find them by solving the equations: Squaring the first and using the second to eliminate y gives $x^4 = x$ and so either $x = 0$ or $x^3 = 1$. The solution to the latter is $x = 1$. From $y = x^2$, we find the corresponding values of y .

- (b) The arc length of the boundary of \mathcal{R} that is on the parabola $y = x^2$.

Ans. $\int_0^1 \sqrt{1 + 4x^2} dx$ or $\int_0^1 \sqrt{1 + 4y^2} dy$

- (c) The volume of \mathcal{V} .

Ans. $\int_0^1 \pi(\sqrt{y^2} - (y^2)^2) dy = \int_0^1 \pi(y - y^4) dy$

- (d) The surface area of \mathcal{V} . *Be careful:* \mathcal{V} has what might be called an inner and outer surface. The surface area is the sum of the areas of these two surfaces.

Ans. $\int_0^1 2\pi\sqrt{y} \sqrt{1 + (1/2y^{1/2})^2} dy + \int_0^1 2\pi y^2 \sqrt{1 + (2y)^2} dy$
 $= \int_0^1 2\pi\sqrt{y + 1/4} dy + \int_0^1 2\pi y^2 \sqrt{1 + 4y^2} dy$

2. (20 pts.) Given the two curves $r = 2$ and $r = 4 \cos \theta$ in polar coordinates.

- (a) Find the polar coordinates of the points where the curves intersect.

Ans. Setting the two values of r equal gives $\cos \theta = 1/2$ and so $\theta = \pm\pi/3$.

- (b) Set up (but *do not evaluate*) an integral for the area that lies inside the curve $r = 4 \cos \theta$ but outside the curve $r = 2$; that is, the area of the region for which $2 \leq r \leq 4 \cos \theta$.

Ans. The curve $r = 2$ lies inside $r = 4 \cos \theta$ for $-\pi/3 < \theta < \pi/3$ and outside it otherwise. Thus the answer is

$$\int_{-\pi/3}^{\pi/3} \left((1/2)(4 \cos \theta)^2 - (1/2)2^2 \right) d\theta = \int_{-\pi/3}^{\pi/3} 2(4 \cos^2 \theta - 1) d\theta,$$

but you need not simplify. Integrating from $5\pi/3$ to $\pi/3$ is NOT correct, but integrating from $5\pi/3$ to $7\pi/3$ is correct. Also, by symmetry, you could have gotten $2 \int_0^{\pi/3} 2(4 \cos^2 \theta - 1) d\theta$,

3. (30 pts.) Express each of the following in the form $a + bi$ with a and b real numbers.

- (a) $(2 + 4i)/(1 - 7i)$

Ans. $\frac{2 + 4i}{1 - 7i} = \frac{(2 + 4i)(1 + 7i)}{(1 - 7i)(1 + 7i)} = \frac{-26 + 18i}{50} = (-13/25) + (9/25)i = -0.52 + 0.36i$

(b) $(2\sqrt{3} + 2i)^{20}$

Ans. Put $2\sqrt{3} + 2i$ in polar form: $r = \sqrt{12 + 4} = 4$ and $\theta = \arctan(1/\sqrt{3}) = \pi/6$.

Raise to the 20th power: $4^{20} e^{20\pi i/6} = 4^{20} e^{4\pi i/3} = 4^{20} \left(-(1/2) - (\sqrt{3}/2)i \right) = -2^{39} - 2^{39}\sqrt{3}i$

(c) $e^{3+i\pi/2}$

Ans. $e^3(\cos(\pi/2) + i \sin(\pi/2)) = 0 + e^3i = e^3i$

4. (20 pts.) (a) Determine whether $\int_0^\infty e^{-x} dx$ is convergent or divergent.

Ans. $\int_0^\infty e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx = \lim_{b \rightarrow \infty} \left(-e^{-x} \Big|_0^b \right) = 1$, so it is convergent

(b) Use part (a) and the comparison theorem to determine whether $\int_0^\infty \frac{e^{-x}}{2 + \sin x} dx$ is convergent or divergent.

Ans. Since $2 + \sin x \geq 1$, the integrand in (b) is less than or equal to the integrand in (a). Hence we have convergence by the comparison theorem.

5. (30 pts.) Evaluate the following integrals.

(a) $\int \frac{\ln x}{x^2} dx$

Ans. Integrate by parts directly or let $\ln x = u$ and integrate by parts.

In the first case, $u = \ln x$ and $dv = x^{-2} dx$. Thus $du = dx/x$ and $v = -1/x$ and so

$$\int \frac{\ln x}{x^2} dx = -(\ln x)/x + \int x^{-2} dx = -(\ln x)/x - (1/x) + C.$$

In the second case, we have $t = \ln x$ and so $dt = dx/x$ and $x = e^t$. Thus $\int \frac{\ln x}{x^2} dx = \int te^{-t} dt$ after some algebra. Let $u = t$ and $dv = e^{-t} dt$ to integrate by parts. We omit the rest.

(b) $\int \cos x \cos(3x) dx$

Ans. Using $\cos A \cos B = [\cos(A - B) + \cos(A + B)]/2$ from Section 7.2 and the fact that $\cos(-C) = \cos C$, we have

$$\int \cos x \cos(3x) dx = \frac{1}{2} \int (\cos(2x) + \cos(4x)) dx = \frac{-\sin(2x)}{4} - \frac{\sin(4x)}{8} + C.$$

(c) $\int \frac{dx}{x^2 \sqrt{1-x^2}}$

Ans. Let $x = \sin t$. Then $dx = \cos t dt$ and so

$$\int \frac{dx}{x^2 \sqrt{1-x^2}} = \int \frac{\cos t dt}{\sin^2 t \cos t} = \int \csc^2 t dt = -\cot t + C.$$

Drawing the right triangle corresponding to $\sin t = x$, we see that $\cot t = \frac{\sqrt{1-x^2}}{x}$ and so our

answer is $\frac{-\sqrt{1-x^2}}{x} + C$

6. (10 pts.) Write out the partial fraction decomposition of the function $\frac{x}{x^2 - 1}$.

Ans. Since $x^2 - 1 = (x - 1)(x + 1)$, we have $\frac{x}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1} = \frac{(A + B)x + (A - B)}{x^2 - 1}$. Thus $A + B = 1$ and $A - B = 0$. Solving gives $A = B = 1/2$ and so $\frac{x}{x^2 - 1} = \frac{1/2}{x - 1} + \frac{1/2}{x + 1}$.

7. (15 pts.) Solve the differential equation $\frac{dy}{dx} = \frac{x^2 + 1}{x^2 y}$ with the initial condition $y(1) = -2$ for y as a *function* of x ; that is, find $y(x)$.

Ans. Separating variables gives $\int y \, dy = \int \frac{x^2 + 1}{x^2} \, dx = \int (1 + x^{-2}) \, dx$ and so $y^2/2 = x - 1/x + C$. By the initial condition, $(-2)^2/2 = 1 - 1 + C$ and so $C = 2$. This is not the complete answer since there are two choices for the square root. Solving and using the fact that $y(1)$ is negative, we have $y = -\sqrt{2x - 2/x + 4}$.

8. (15 pts.) Use Euler's method with step size 1.0 to estimate $x(3.0)$, where $x(t)$ is the solution of the initial value problem

$$\frac{dx}{dt} = x + t, \quad x(0) = 0.$$

Ans. We have $h = 1$. The following computation gives the answer 4.

n	t_n	x_n	$h(dx/dt)$
0	0	0	0
1	1	0	1
2	2	1	3
3	3	4	

9. (20 pts.) Consider the integral $I = \int_0^2 \frac{2}{4x + 1} \, dx$.

(a) Use the Midpoint rule with $n = 2$ subintervals to approximate I .

Ans. Let $f(x) = 2(4x + 1)^{-1}$. We have $\Delta x = (2 - 0)/2 = 1$. Thus the Midpoint rule gives $1(f(1/2) + f(3/2)) = 2/(4/2 + 1) + 2/(4 \times 3/2 + 1) = 2/3 + 2/7 = 20/21$.

(b) How large should n be so that the midpoint approximation of I is accurate to within 6×10^{-4} ?

Ans. We need to bound $|f''(x)|$ for $0 \leq x \leq 2$. We have $f'(x) = -8(4x + 1)^{-2}$ and $f''(x) = 64(4x + 1)^{-3}$. Since $1 \leq 4x + 1 \leq 5$, we have $64 \geq f''(x) \geq 64/5^3$. Thus we can take $K = 64$ in the formula $E_M \leq K(b - a)^3/24n^2$. This gives $E_M \leq 64 \times 2^3/24n^2$ and so we need $64 \times 8/24n^2 \leq 6 \times 10^{-4}$. This gives $n^2 \geq \frac{64 \times 8 \times 10^4}{24 \times 6} = \frac{32 \times 10^4}{9}$. Hence

$$n \geq \sqrt{\frac{32 \times 10^4}{9}} = \frac{400}{\sqrt{3}}.$$